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CONSERVATIVE DIFFERENCE SCHEMES FOR OPTIMAL PLACEMENT OF HEAT SOURCES IN A PARALLELEPIPED

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Abstract—In this paper, a method and algorithm for solving the non-stationary problem of optimal placement of heat sources of minimum power in the space have been developed. As a result, the temperature in the space is within the specified limits and the value of the functional is minimized. The mathematical model of the process is described by the heat conduction equation with a variable coefficient. The numerical model of the problem is constructed using implicit conservative difference schemes. To solve the problem of thermal conductivity control, a linear programming problem was used. Software for numerical modeling has been developed. The results of a computational experiment are presented.

Keywords—optimal placement; heat sources; integro-interpolation method; conservative schemes; Big M method.

I INTRODUCTION

One of the most common objects in various fields of human activity is the system of heat sources, the heat balance in heated rooms. Mathematical modeling of such systems poses the problem of optimal placement of heat sources in heated rooms, which is associated with resource-saving engineering technologies. The task of optimal placement of heat sources in heated areas has always been relevant in design work in construction, greenhouses and other technical and technological areas.

The heat transfer process can be controlled in different ways. The process is often controlled by the placement of heat sources or changes in ambient temperature. The problems of controlling the process of heat propagation under various conditions were studied by A.G.Butkovsky [1], J.L.Lions [2], Yu.V.Egorov [3], A.I.Egorov [4], as well as by other authors, and important results were obtained. Their work forms the basis of this work. In the work [5] the problem of optimal control of processes described by the heat equation was studied. The control parameter is set in the boundary condition and has reached the minimum of the functional given by the integral quadratic expression. A method for finding an admissible control that gives a minimum to the functional is shown. In the paper [6] the third boundary value problem of parabolic type was considered. The right side of the boundary condition contains controls in additive form. The problem of transferring an object from the initial state to the zero state in a conflict situation is solved.

In the work [7], the differential-difference problem of controlling the diffusion process was studied, an analogue of the maximum principle was obtained, which makes it possible to determine the moments of switching on and off the source of maximum power. The paper [8] proposes a solution to the problem of optimal placement of sources in inhomogeneous media, in which scalar stationary fields are described by elliptic equations. The algorithms for solving the problem are based on methods for estimating the values of the functional on the set of possible locations of sources, which makes it possible to choose the optimal variant by implementing the branch and bound method. The paper [9] considers the problem of optimizing the density of heat sources in stationary processes described by elliptic equations given by the third boundary condition. In the work [10], the minimax problem of the optimal placement of sources in the processes of heat propagation described by equations of elliptic type was numerically solved. To find the extremum of the objective function depending on the location of the sources of the physical field, the minimax method of mathematical modeling was used. The proposed approach made it possible to find a numerical solution to the boundary value problem in terms of the source carrier placement parameters. In works [11, 12, 13], a method for the numerical solution of the non-stationary problem of optimal placement of heat sources with a minimum power in processes described by parabolic type equations is proposed. An algorithm and a set of programs for the numerical solution of non-stationary problems of optimal control of the location of heat sources and visualization of the results obtained have been developed.

In the work [14], the problems of optimal space heating based on the Pontryagin maximum principle are considered. The paper [15] considers the problem of energy-efficient heat supply of a building in a central heating system.

In this paper, we consider the problem of heat conduction control based on the optimization of a linear objective functional, taking into account constraints, which is solved on the basis of approximation and reduction to a linear programming problem. The paper proposes a technique and algorithm for solving the non-stationary problem of maintaining the temperature inside the region within the given limits, by optimally placing heat sources in a parallelepiped. Software was developed for carrying out computational experiments.

II Statement of the problem and its conservative approximation

In the domain $D=\{a\le x\le b,\ c\le y\le d,\ p\le z\le q,\ 0\le t\le T\}$, it is required to find a function $f(x,y,z,t)\ge 0$ such that for any t the linear functional

$$J\{f\}=\int_a^b\int_c^d\int_p^q f(x,y,z,t)\,dzdydx\to\min, \tag{1}$$

reaches a minimum and satisfies the following conditions:

$$\begin{gathered}\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\chi\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\chi\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\chi\frac{\partial u}{\partial z}\right)+\\ f,\quad x\in(a,b),\ y\in(c,d),\ z\in(p,q),\ t\in(0,T],\\ u(x,y,z,0)=u_0(x,y,z),\\ u(a,y,z,t)=\mu_1(y,z,t),\quad u(b,y,z,t)=\mu_2(y,z,t),\\ u(x,c,z,t)=\mu_3(x,z,t),\quad u(x,d,z,t)=\mu_4(x,z,t),\\ u(x,y,p,t)=\mu_5(x,y,t),\quad u(x,y,q,t)=\mu_6(x,y,t),\end{gathered} \tag{2}$$

$$m(x,y,z,t)\le u(x,y,z,t)\le M(x,y,z,t),\ (x,y,z,t)\in D, \tag{3}$$

where $u=u(x,y,z,t)$ is the temperature at the point (x,y,z) of the parallelepiped at time t; $\chi=\chi(x,y,z)$ is the thermal conductivity coefficient; $u_0(x,y,z)$, $\mu_1(y,z,t)$, $\mu_2(y,z,t)$, $\mu_3(x,z,t)$, $\mu_4(x,z,t)$, $\mu_5(x,y,t)$, $\mu_6(x,y,t)$, $m(x,y,z,t)$, $M(x,y,z,t)$ are given functions. The functions $m(x,y,z,t)$ and $M(x,y,z,t)$ are the minimum and maximum temperatures defined in the domain D. $f=f(x,y,z,t)$ is the heat source defined in the space $L_2(D)$.

Let $Lu=\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\chi(x,y,z)\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(\chi(x,y,z)\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial z}\left(\chi(x,y,z)\frac{\partial u}{\partial z}\right)$. The operator L defined in $L_2(D)$ has an inverse L^{-1}. Here L^{-1} is an integral operator with a continuous kernel (Green's function). Using it, we can write problem (1)-(3) in the following form:

$$\begin{gathered}m(x,y,z,t)\le(L^{-1}f)(x,y,z,t)\le M(x,y,z,t),\\ f(\cdot,\cdot,\cdot,\cdot)\in L_2(D),\quad f(x,y,z,t)\ge 0.\end{gathered} \tag{4}$$

We will solve this problem in a complete mathematical formulation by the integro-interpolation method on a uniform grid.

Introduce in D a difference grid uniform in four variables $\overline{\omega}^{\tau}_{h_1h_2h_3}=\overline{\omega}_{h_1}\times\overline{\omega}_{h_2}\times\overline{\omega}_{h_3}\times\overline{\omega}^{\tau}=\{(x_i,y_j,z_k,t_s):\ x_i=ih_1,\ y_j=jh_2,\ z_k=kh_3,\ t_s=s\tau,\ i=\overline{0,N_1},\ j=\overline{0,N_2},\ k=\overline{0,N_3},\ s=\overline{0,N_4}\}$ with steps $h_1=(b-a)/N_1,\ h_2=(d-c)/N_2,\ h_3=(q-p)/N_3,\ \tau=T/N_4$.

To obtain conservative difference schemes, we use the integro-interpolation method. To obtain a difference equation, we write an integral heat balance equation on a parallelepiped $x_{i-1/2}\le x\le x_{i+1/2}$, $y_{j-1/2}\le y\le y_{j+1/2}$, $z_{k-1/2}\le z\le z_{k+1/2}$ for time $t_s\le t\le t_{s+1}$ [16]:

$$\begin{gathered}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{z_{k-1/2}}^{z_{k+1/2}}(u(x,y,z,t_{s+1})-u(x,y,z,t_s))dzdydx=\\ \int_{t_s}^{t_{s+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{z_{k-1/2}}^{z_{k+1/2}}(W(x_{i-1/2},y,z,t)-W(x_{i+1/2},y,z,t))dzdydt+\\ \int_{t_s}^{t_{s+1}}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{z_{k-1/2}}^{z_{k+1/2}}(W(x,y_{j-1/2},z,t)-W(x,y_{j+1/2},z,t))dzdxdt+\\ \int_{t_s}^{t_{s+1}}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}(W(x,y,z_{k-1/2},t)-W(x,y,z_{k+1/2},t))dydxdt+\\ \int_{t_s}^{t_{s+1}}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{z_{k-1/2}}^{z_{k+1/2}}f(x,y,z,t)dzdydxdt.\end{gathered}$$

Here $W(x,y,z,t)$ is the heat flux, $W(x,y,z,t)=-\chi(x,y,z)\,\mathrm{grad}\,u$.

We approximate the integrals included in the balance equation by approximate formulas

$$\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{z_{k-1/2}}^{z_{k+1/2}}u(x,y,z,t_{s+1})dzdydx\approx h_1h_2h_3u^{s+1}_{ijk},$$

$$\int_{t_s}^{t_{s+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{z_{k-1/2}}^{z_{k+1/2}}W(x_{i-1/2},y,z,t)dzdydt\approx\tau h_2h_3W^{s+1}_{i-1/2jk},$$

$$\int_{t_s}^{t_{s+1}}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{z_{k-1/2}}^{z_{k+1/2}}W(x,y_{j-1/2},z,t)dzdxdt\approx\tau h_1h_3W^{s+1}_{ij-1/2k},$$

$$\int_{t_s}^{t_{s+1}}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}} W(x,y,z_{k-1/2},t)dydxdt \approx \tau h_1 h_2 W^{s+1}_{ijk-1/2},$$

$$\int_{t_s}^{t_{s+1}}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{z_{k-1/2}}^{z_{k+1/2}} f(x,y,z,t)dzdydxdt \approx \tau h_1 h_2 h_3 f^{s+1}_{ijk},$$

$$W^{s+1}_{i-1/2jk} = -\chi_{i-1/2jk}\frac{u^{s+1}_{ijk}-u^{s+1}_{i-1jk}}{h_1},$$

$$W^{s+1}_{ij-1/2k} = -\chi_{ij-1/2k}\frac{u^{s+1}_{ijk}-u^{s+1}_{ij-1k}}{h_2},$$

$$W^{s+1}_{ijk-1/2} = -\chi_{ijk-1/2}\frac{u^{s+1}_{ijk}-u^{s+1}_{ijk-1}}{h_3}.$$

In this case, $\chi_{i-1/2jk}$, $\chi_{ij-1/2k}$, $\chi_{ijk-1/2}$ and f^{s+1}_{ijk} are defined by the equalities

$$\chi_{i-1/2jk} = \chi\left(\frac{x_i+x_{i-1}}{2}, y_j, z_k\right),$$

$$\chi_{ij-1/2k} = \chi\left(x_i, \frac{y_j+y_{j-1}}{2}, z_k\right),$$

$$\chi_{ijk-1/2} = \chi\left(x_i, y_j, \frac{z_k+z_{k-1}}{2}\right),$$

$$\chi_{ijk} = \chi(x_i,y_j,z_k), \quad f^{s+1}_{ijk} = f(x_i,y_j,z_k,t_{s+1}).$$

The implicit conservative difference scheme for problem (2) has the form:

$$\begin{cases} \frac{u^{s+1}_{ijk}-u^{s}_{ijk}}{\tau} = \Bigg[\chi_{i+1/2jk}\frac{u^{s+1}_{i+1jk}-u^{s+1}_{ijk}}{h_1^2} - \\ \chi_{i-1/2jk}\frac{u^{s+1}_{ijk}-u^{s+1}_{i-1jk}}{h_1^2}\Bigg] + \Bigg[\chi_{ij+1/2k}\frac{u^{s+1}_{ij+1k}-u^{s+1}_{ijk}}{h_2^2} - \\ \chi_{ij-1/2k}\frac{u^{s+1}_{ijk}-u^{s+1}_{ij-1k}}{h_2^2}\Bigg] + \Bigg[\chi_{ijk+1/2}\frac{u^{s+1}_{ijk+1}-u^{s+1}_{ijk}}{h_3^2} - \\ \chi_{ijk-1/2}\frac{u^{s+1}_{ijk}-u^{s+1}_{ijk-1}}{h_3^2}\Bigg] + f^{s+1}_{ijk},\ i=\overline{1,N_1-1}, \\ j=\overline{1,N_2-1},\ k=\overline{1,N_3-1},\ s=\overline{0,N_4-1}, \\ u^0_{ijk} = u_0(x_i,y_j,z_k), \\ u^{s+1}_{0jk} = \mu_1(y_j,z_k,t_{s+1}),\ u^{s+1}_{N_1jk} = \mu_2(y_j,z_k,t_{s+1}), \\ u^{s+1}_{i0k} = \mu_3(x_i,z_k,t_{s+1}),\ u^{s+1}_{iN_2k} = \mu_4(x_i,z_k,t_{s+1}), \\ u^{s+1}_{ij0} = \mu_5(x_i,y_j,t_{s+1}),\ u^{s+1}_{ijN_3} = \mu_6(x_i,y_j,t_{s+1}), \\ i=\overline{0,N_1},\ j=\overline{0,N_2},\ k=\overline{0,N_3},\ s=\overline{0,N_4-1}. \end{cases} \tag{5}$$

Let us introduce the notation

$$\overline{XYZ} = \left(\frac{1}{\tau} + \frac{\chi_{i\pm1/2jk}}{h_1^2} + \frac{\chi_{ij\pm1/2k}}{h_2^2} + \frac{\chi_{ijk\pm1/2}}{h_3^2}\right),$$

$$X^+ = -\frac{\chi_{i+1/2jk}}{h_1^2}, \quad X^- = -\frac{\chi_{i-1/2jk}}{h_1^2}, \quad Y^+ = -\frac{\chi_{ij+1/2k}}{h_2^2},$$

$$Y^- = -\frac{\chi_{ij-1/2k}}{h_2^2}, \quad Z^+ = -\frac{\chi_{ijk+1/2}}{h_3^2}, \quad Z^- = -\frac{\chi_{ijk-1/2}}{h_3^2}.$$

Consider the matrix

$$A = \begin{bmatrix} \overline{XYZ} & Z^+ & 0 & \dots & 0 & Y^+ & 0 & \dots & 0 & X^+ & 0 & \dots & \dots & 0 \\ Z^- & \overline{XYZ} & Z^+ & 0 & \dots & 0 & Y^+ & 0 & \dots & 0 & X^+ & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & X^- & 0 & \dots & 0 & Y^- & 0 & \dots & 0 & Z^- & \overline{XYZ} & Z^+ \\ 0 & \dots & \dots & 0 & X^- & 0 & \dots & 0 & Y^- & 0 & \dots & 0 & Z^- & \overline{XYZ} \end{bmatrix}.$$

We get

$$G = A^{-1}.$$

We approximate problem (1)-(5) in the form of a linear programming problem. We divide the region D by x,y,z,t into N_1, N_2, N_3, N_4 equal parts, respectively: $D = \bigcup_{s=1}^{N_4}\bigcup_{i=1}^{N_1}\bigcup_{j=1}^{N_2}\bigcup_{k=1}^{N_3} D^s_{ijk}$, where $D^s_{ijk} = \{(x,y,z,t),\ x_{i-1} \le x \le x_i,$ $y_{j-1} \le y \le y_j,\ z_{k-1} \le z \le z_k,\ t_{s-1} \le t \le t_s\}$, $i=\overline{1,N_1}$, $j=\overline{1,N_2}$, $k=\overline{1,N_3}$, $s=\overline{1,N_4}$. In the space $L_2(D)$, the functions $f(x,y,z,t) = f^s_{ijk}$, $(x,y,z,t) \in D^s_{ijk}$ $(i=\overline{1,N_1-1}$, $j=\overline{1,N_2-1}$, $k=\overline{1,N_3-1}$, $s=\overline{1,N_4})$ are defined as piecewise constant functions. From here we get $f(x,y,z,t) \approx \sum_{s=1}^{N_4}\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} f^s_{ijk}$.

Let $g_{rw} = G$, $m^s_{ijk} = m(x_i,y_j,z_k,t_s)$, $M^s_{ijk} = M(x_i,y_j,z_k,t_s)$, $\tilde{f}^s_w = f^s_{ijk}$, $r=w$, $w=(i-1)(N_2-1)(N_3-1)+(j-1)(N_3-1)+k$, $N=(N_1-1)(N_2-1)(N_3-1)$, $r=\overline{1,N}$, $i=\overline{1,N_1-1}$, $j=\overline{1,N_2-1}$, $k=\overline{1,N_3-1}$, $s=\overline{1,N_4}$. We substitute the expression $f(x,y,z,t)$ into (1) and replace inequality (4) with grid functions.

After that, we get the following linear programming problem:

$$\begin{gathered} J_s\{f\} = \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}(\mathrm{mes}D^s_{ijk})f^s_{ijk} \to \min, \\ m^s_{ijk} \le \sum_{w=1}^{N} g_{rw}\tilde{f}^s_w \le M^s_{ijk}, \quad r=1,2,\dots,N, \\ i=\overline{1,N_1-1},\ j=\overline{1,N_2-1},\ k=\overline{1,N_3-1},\ s=\overline{1,N_4}, \\ \tilde{f}^s_w \ge 0, \quad w=1,2,\dots,N,\ s=1,2,\dots,N_4. \end{gathered} \tag{6}$$

Problem (6) is solved by the big M method [17, 18]. The numerical solution of problem (2) is found using $u^s_{ijk} = \sum_{w=1}^{N} g_{rw}\tilde{f}^s_w$. The found $\tilde{f}^s_w$ is a function that gives a minimum to the functional (1).

III DESCRIPTION OF THE ALGORITHM AND MODELING RESULTS

For an approximate solution of problem (1)-(6), software in the *C#* language has been developed. It allows you to represent all the necessary input data: constants, coefficients, grid parameters, as well as temperature functions, initial and boundary conditions, in the form of scripts. Graphical modules have been developed to present the results.

The flowchart (Fig. 1) shows a general algorithm for the numerical solution of the non-stationary problem of controlling the optimal location of heat sources.

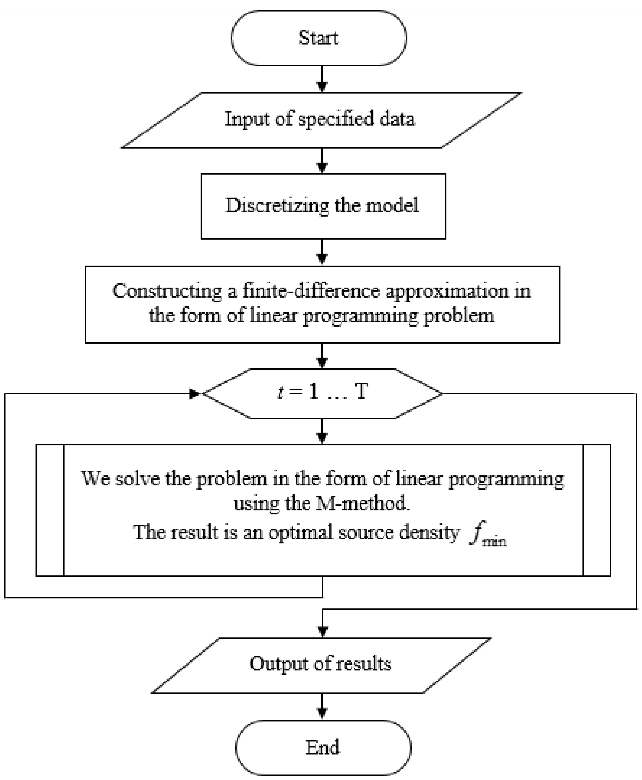

Fig. 1: Flowchart of the general algorithm for solving the problem

Computational experiment. Find the optimal location of heat sources with the minimum power cubed. The problem was solved with the following values of input parameters: $x, y, z \in [0, 1]$, thermal diffusivity $\chi(x, y, z) = x^2y^2z^2$ m^2/s, the initial and boundary conditions are determined by the functions: $u_0(x, y, z) = 2 + x^2 + y^2 + z^2$ m/s, $\mu_1(y, z, t) = 2 + y^2 + z^2 + t^2$ m/s, $\mu_2(y, z, t) = 3 + y^2 + z^2 + t^2$ m/s, $\mu_3(x, z, t) = 2 + x^2 + z^2 + t^2$ m/s, $\mu_4(x, z, t) = 3 + x^2 + z^2 + t^2$ m/s, $\mu_5(x, y, t) = 2 + x^2 + y^2 + t^2$ m/s, $\mu_6(x, y, t) = 3 + x^2 + y^2 + t^2$ m/s, the minimum and maximum temperatures are given by the functions $m(x, y, z, t) = 1 + x^2 + y^2 + z^2 + t^2$ K, $M(x, y, z, t) = 4 + x^2 + y^2 + z^2 + t^2$ K, end of time $T = 1$. Computational grid with the number of sources $(N_1 - 1) \times (N_2 - 1) \times (N_3 - 1) \times N_4 = 6 \times 6 \times 6 \times 7$. The minimum value of the functional in the numerical solution is $J_{\min} = 14.35$ K·m/s. On fig. 2 presents the results of the numerical solution of problem (6). Results are shown with minimum (borders in blue, below), maximum (borders in red, above), and approximate (green, middle) temperature values. On fig. 3 shows the optimal location of heat sources with a minimum power in the form of a bar graph.

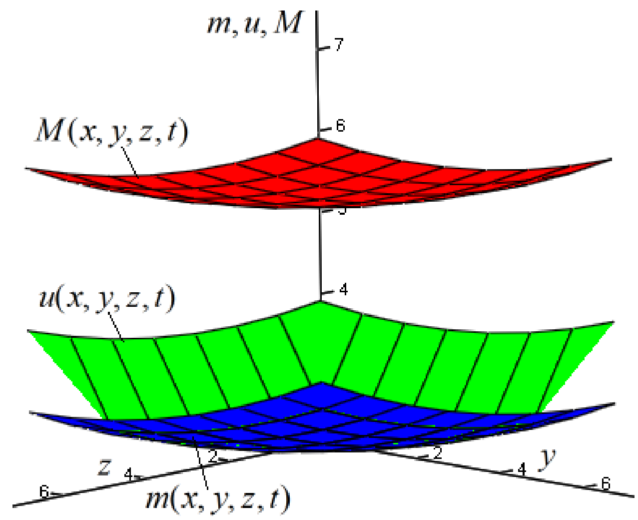

Fig. 2: Graph of the solution of problem (6) at $x = 0.5$, $t = T$

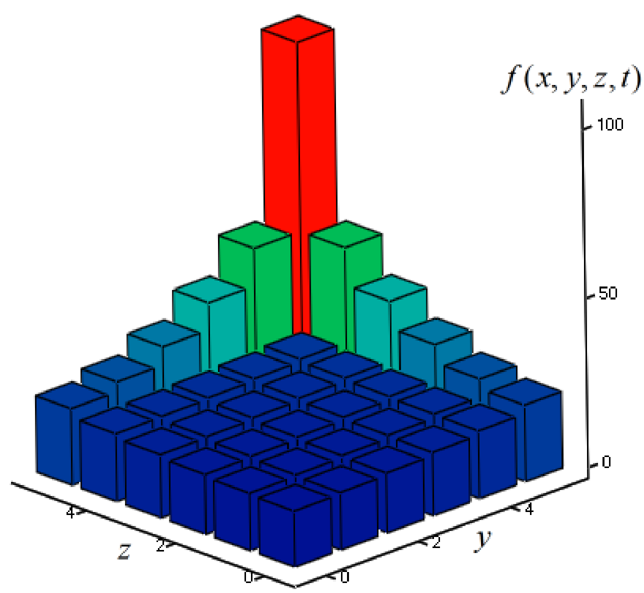

Fig. 3: Optimal placement of heat sources $f(x, y, z, t)$ at $x = 0.5$, $t = T$

IV Conclusion

As you know, the construction of the Green's function for problems in partial derivatives, in fact, means finding a solution in an explicit form. When applying numerical methods, the values of the Green's function are presented in the form of a matrix, which is inverse to the matrix composed of the coefficients of the system of linear algebraic equations. Thus, it is possible to indicate the values of the desired function at the nodal points of the partition. After substituting these values for the conditions-restrictions of body temperature, taking into account finding the extremum of the functional, a linear programming problem is obtained, for the solution of which the standard algorithm big M method is used. A technique and algorithm for solving the non-stationary problem of ensuring the temperature inside the region within the given limits by optimal placement of heat sources in a parallelepiped are proposed. The results of this computational experiment show that the functional reaches its minimum.

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