*Acta of Turin Polytechnic University in Tashkent*, 2022, 31, 28-33
Published Online March 2022 in Acta TTPU (http://www.acta.polito.uz/)

# ON SOME PROPERTIES OF CONTINUED FRACTIONS AND RETURN TIME FOR CIRCLE HOMEOMORPHISMS

**J.J.Karimov[1], A.N.Qodiriy[2] and S.J.Jahongirov[3]**
Turin Polytechnic University in Tashkent
[1]Email: jkarimov0702@gmail.com
[2]Email: abduraufqodiriy@gmail.com
[3]Email: samandarjahongirov574@gmail.com

**Abstract**—In present work we study general properties of continued fractions and the return times for circle homeomorphisms with irrational rotation number. Consider the set $X$ of all orientation preserving circle homeomorphisms $T$ with one break point and irrational rotation number. There are given proof of the main theorem for return time using visualizations and constructed example to computing return time for irrational rotation number.

**Keywords**—circle homeomorphism, break point, rotation number, continued fractions, return time.

## I Introduction

This paper is devoted to study general properties of continued fractions and return times for circle homeomorphisms with irrational rotation number in dynamical partitions. Continued Fractions are important in many branches of mathematics. They arise naturally in long division and in the theory of approximation to real numbers by rationals. These objects that are related to number theory help us find good approximations for real life constants. In A. Khinchin's classic book on continued fractions [1], he defines two notions of being a "best approximation" to a number. The first is the easier one to describe: a fraction $c/d$ is a best approximation to a number a if $c/d$ is closer to a than any number with a smaller denominator. That is, if $|a-c/d| < |a-p/q|$ for any other fraction $p/q$ where $q < d$. Khinchin calls this a *best approximation of the first kind*. The fraction $c/d$ is a *best approximation of the second kind* for a number a if for every other fraction $p/q$ with $q < d$, $|da-c| < |qa-p|$. It's a similar relation as the first kind, but we multiply through by the denominator on both sides. All best approximations of the second kind are best approximations of the first kind, but not all best approximations of the first kind are best approximations of the second kind. Convergents of the continued fraction for a number are best approximations of the second kind, and they're the only numbers that are best approximations of the second kind.

One of the important problems of ergodic theory is to study the behaviour of return times. D.H.Kim and B.K.Seo in [2] investigated return time and waiting time for partition $Q_n$ of same first $n$ digits in binary expansion, i.e. $Q_n = \{[0,2^{-n}),\dots,[1-2^{-n},1)\}$. We consider return time in more general partition, which is called *dynamical partition* (See Section 3). Let $(X,\mathbb{B},\mu)$ be a probability measure space and $T: X > X$ be be an orientation preserving homeomorphism of the circle $S^1=\mathbb{R}^1/\mathbb{Z}^1 \simeq [0,1)$ with irrational rotation number $\theta$. Let $\mu$ be the unique invariant probability measure of T. Consider the measurable subset $E \subset X$, $\mu(E) > 0$ and a point $x \in X$ which returns to $E$ under iterations by $T$, we define first return time $R_E$ on $E$ by the following way:

$$R_E(x) = min\{j \geq 1 : T^j x \in E\}.$$

Kac's lemma [3] states that $\int_E R_E(x)d\mu \leq 1$. If $T$ is ergodic, then the equality holds.

A.Dzhalilov and J.Karimov studied the entrance times for circle homeomorphisms with one break point and "golden mean" rotation number ($\rho = [1,1,\dots,1,\dots] = \frac{\sqrt{5}-1}{2}$) and universal renormalization properties [4].

## II Continued Fractions. Proof of general properties

In this section we prove the general properties of continued fractions for irrational number.

1. Let

$$\theta = [a_0,a_1,a_2,\dots,a_n] = a_0 + \frac{1}{a_1+\frac{1}{a_2+\frac{1}{\dots+\frac{1}{a_n}}}},\ n \in N$$

and
$$\frac{p_k}{q_k} = [a_0, a_1, a_2, \ldots, a_k],\ 0 \le k \le n,\ k \in N.$$

The following equations are satisfied for all $i$ such that $0 \le i \le n$, $i \in N$:

$$p_i = a_i p_{i-1} + p_{i-2},\ q_i = a_i q_{i-1} + q_{i-2}.$$

We also have $p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0$.

**Proof.** Let $S_{-1} = 1$ and $S_0 = a_n$:

$$[a_{n-1}, a_n] = a_{n-1} + \frac{1}{a_n} = \frac{a_{n-1}a_n + 1}{a_n} = \frac{a_{n-1}S_0 + S_{-1}}{S_0}.$$

Let $S_1 = a_{n-1}S_0 + S_{-1}$. Then we will have

$$[a_{n-1}, a_n] = \frac{S_1}{S_0}.$$

We can see that

$$S_k = a_{n-k}S_{k-1} + S_{k-2}, \tag{1}$$

and
$$[a_{n-k}, a_{n-k+1}, \ldots, a_n] = \frac{S_k}{S_{k-1}}.$$

We have $\theta = [a_0, a_1, \ldots, a_n] = [a_{n-n}, a_{n-n+1}, \ldots, a_n]$. Thus we obtain $k = n$. Then $\theta = \frac{S_n}{S_{n-1}}$. We know that $\theta = \frac{p_n}{q_n}$. From that
$$\frac{p_n}{q_n} = \frac{S_n}{S_{n-1}}$$

We cannot say $p_n$ is the same as $S_n$ because of $S_0 = a_n$ and $p_0 = a_0$ are not equal at the all time.

Let's do some substitutions on $S_n$. From the equation (1) we obtain

$$S_n = a_0 S_{n-1} + S_{n-2} = a_0(a_1 S_{n-2} + S_{n-3}) + S_{n-2} =$$

$$= (a_1 a_0 + 1)S_{n-2} + a_0 S_{n-3}$$

Let $T_{-1} = 1$ and $T_0 = a_0$. Then

$$S_n = (a_1 T_0 + T_{-1})S_{n-2} + T_0 S_{n-3}$$

Let $T_1 = a_1 T_0 + T_{-1}$. Then

$$S_n = T_1 S_{n-2} + T_0 S_{n-3}$$

Therefore

$$S_n = T_1(a_2 S_{n-3} + S_{n-4}) + T_0 S_{n-3} = (a_2 T_1 + T_0)S_{n-3} + T_1 S_{n-4}$$

Similarly consider $T_2 = a_2 T_1 + T_0$, then:

$$S_n = T_2 S_{n-3} + T_1 S_{n-4}$$

We can see that
$$T_k = a_k T_{k-1} + T_{k-2}$$

and
$$S_n = T_{k-1}S_{n-k} + T_{k-2}S_{n-k-1}.$$

If $n = k$ then

$$S_n = T_{n-1}S_0 + T_{n-2}S_{-1} = T_{n-1}a_n + T_{n-2}$$

But it is known that $T_n = a_n T_{n-1} + T_{n-2}$. Thus $S_n = T_n$. Therefore $p_n = T_n$. If we look at $p_0 = a_0$ and $T_0 = a_0$, we can obtain that they are equal. So, $p_n$ is the sequence as same as the sequence $T_n$. Then we can conclude that $p_n$ also has the same recurrence equation:

$$p_i = a_i p_{i-1} + p_{i-2}$$

We can prove $q_i = a_i q_{i-1} + q_{i-2}$ using above method. In this case, we should substitute $S_1$ instead of $S_0$.

2. For all $n \in N$, following equality holds:

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$$

**Proof.** We use the 1st property and obtain

$$p_n q_{n-1} - p_{n-1} q_n = (a_n p_{n-1} + p_{n-2})q_{n-1} - p_{n-1}(a_n q_{n-1} + q_{n-2}) =$$

$$= a_n p_{n-1} q_{n-1} + p_{n-2} q_{n-1} - a_n p_{n-1} q_{n-1} - p_{n-1} q_{n-2} =$$

$$= p_{n-2} q_{n-1} - p_{n-1} q_{n-2} = (-1)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}).$$

We obtain

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^k (p_{n-k} q_{n-k-1} - p_{n-k-1} q_{n-k}).$$

Let $k = n + 1$.

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}(p_{-1} q_{-2} - p_{-2} q_{-1}).$$

It is known that $p_{-2} = 0,\ p_{-1} = 1,\ q_{-2} = 1,\ q_{-1} = 0$. Thus

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

3. For all $n \in N$ the following equality holds:

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

**Proof.**

$$p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1} + p_{n-2})q_{n-2} - p_{n-2}(a_n q_{n-1} + q_{n-2}) =$$

$$= a_n p_{n-1} q_{n-2} + p_{n-2} q_{n-2} - a_n p_{n-2} q_{n-1} - p_{n-2} q_{n-2} =$$

$$= a_n p_{n-1} q_{n-2} - a_n p_{n-2} q_{n-2} = a_n(p_{n-1} q_{n-2} - p_{n-2} q_{n-2}).$$

From the 2nd property, it is known that

$$p_{n-1} q_{n-2} - p_{n-2} q_{n-2} = (-1)^{n-2} = (-1)^n$$

So,

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$$

Before starting the proof of the next properties, consider some important notations:

a) Let $k$ be non-negative number such that $0 \leq k \leq n$:

$$a'_k = [a_k, a_{k+1}, \ldots, a_n]$$

So, we can easily see that

$$\theta = [a_0, a_1, \ldots, a_{k-1}, a'_k]$$

Also, it is necessary to show the following equations:

$$p'_k = a'_k p_{k-1} + p_{k-2} \quad (2)$$

$$q'_k = a'_k q_{k-1} + q_{k-2} \quad (3)$$

Thus, we can write $\theta$ as:

$$\theta = \frac{p'_k}{q'_k}$$

b) For all $x \in R$, we define the distance to the nearest integer as following:

$$\|x\| = \min_{n \in Z} |x - n|$$

From that we can say that $\|x\| \in [0; 0.5]$

4. For all $i \in N$ such that $0 \leq i \leq n$:

$$\theta - \frac{p_i}{q_i} = \frac{(-1)^i}{q_i q'_{i+1}}$$

**Proof.** Let

$$\theta = \frac{p'_{i+1}}{q'_{i+1}}$$

Thus,

$$\theta - \frac{p_i}{q_i} = \frac{p'_{i+1}}{q'_{i+1}} - \frac{p_i}{q_i} = \frac{p'_{i+1} q_i - p_i q'_{i+1}}{q_i q'_{i+1}}$$

By the equations (2) and (3), we can write the equality above as following:

$$\theta - \frac{p_i}{q_i} = \frac{(a'_{i+1} p_i + p_{i-1}) q_i - p_i (a'_{i+1} q_i + q_{i-1})}{q_i q'_(i+1)} =$$

$$= \frac{a'_{i+1} p_i q_i + p_{i-1} q_i - a'_{i+1} p_i q_i - p_i q_{i-1}}{q_i q'_{i+1}} = -\frac{p_i q_{i-1} - p_{i-1} q_i}{q_i q'_{i+1}}$$

We use the 2nd property:

$$\theta - \frac{p_i}{q_i} = -\frac{(-1)^{i-1}}{q_i q'_{i+1}} = \frac{(-1)^i}{q_i q'_{i+1}}$$

So,

$$\theta - \frac{p_i}{q_i} = \frac{(-1)^i}{q_i q'_{i+1}}$$

We can conclude that the signs of the sequence $\{\theta - p_i/q_i\}_{i=0}^{\infty}$ alternate.

5. For all $i \in N$ such that $1 \leq i \leq n$:

$$\frac{1}{q_{i+1} + q_i} < \|q_i \theta\| < \frac{1}{q_{i+1}}$$

**Proof.** Let's consider previous property:

$$\theta - \frac{p_i}{q_i} = \frac{(-1)^i}{q_i q'_{i+1}} \Longrightarrow |\theta - \frac{p_i}{q_i}| = \frac{1}{q_i q'_{i+1}}$$

We know $q'_{i+1} > q_{i+1}$ and if $i \geq 2$, then $q_i \geq 2$. Then

$$|\theta - \frac{p_i}{q_i}| < \frac{1}{q_i q_{i+1}} \Longrightarrow |q_i \theta - p_i| < \frac{1}{q_{i+1}} \leq 1/2$$

Because of that, we can say $|q_i \theta - p_i| = \|q_i \theta\|$. Thus

$$\|q_i \theta\| < \frac{1}{q_{i+1}}$$

It is known $|q_i \theta - p_i| = 1/q'_{i+1}$. Now we prove that

$$\frac{1}{q'_{i+1}} > \frac{1}{q_{i+1} + q_i}$$

Then

$$\frac{1}{q'_{i+1}} > \frac{1}{q_{i+1} + q_i} \Longrightarrow q'_{i+1} < q_{i+1} + q_i \Longrightarrow$$

$$\Longrightarrow a'_{i+1} q_i + q_{i-1} < a_{i+1} q_i + q_{i-1} + q_i \Longrightarrow a'_{i+1} < a_{i+1} + 1$$

We have

$$a'_{i+1} = a_{i+1} + \frac{1}{a'_{i+2}}$$

Thus

$$a'_{i+1} < a_{i+1} + 1 \Longrightarrow a_{i+1} + \frac{1}{a'_{i+2}} < a_{i+1} + 1 \Longrightarrow a'_{i+2} > 1.$$

The last inequality is true for all $1 \leq i < n$.

6. For all $i \in N$ such that $1 \leq i < n$, if $k \in N$ such that $0 < k < q_{i+1}$, then:

$$\|k\theta\| > \|q_i \theta\|$$

**Proof.** Let $\|k\theta\| = |k\theta - l|$. Then we have:

$$|k\theta - l| > |q_i \theta - p_i|$$

We can set up the following equation system [5]. Here, $x$ and $y$ are some variables:

$$\begin{cases} p_{i+1}x+p_iy=l \\ q_{i+1}x+q_iy=k \end{cases} \Longrightarrow \begin{cases} x=\frac{lq_i-kp_i}{p_{i+1}q_i-p_iq_{i+1}} \\ y=\frac{lq_{i+1}-kp_{i+1}}{p_{i+1}q_i-p_{i+1}q_i} \end{cases}$$

By the 2nd property:

$$\begin{cases} x=(-1)^i(lq_i-kp_i) \\ y=(-1)^i(lq_{i+1}-kp_{i+1}) \end{cases}$$

$x$ and $y$ cannot be zero. Since $l$,$k$ are natural numbers then $x$ and $y$ are integer numbers. Thus $|x|,|y|\geq 1$. We have $q_{i+1}>k$. Then $x$ and $y$ should have opposite signs. Also using the 4th property, $\theta-p_i/q_i$ and $\theta-p_{i+1}/q_{i+1}$ have opposite signs. Thus, $x(q_{i+1}\theta-p_{i+1})$ and $y(q_i\theta-p_i)$ have same sign. Then

$$k\theta-l=(q_{i+1}x+q_iy)\theta-(p_{i+1}x+p_iy)=$$

$$=x(q_{i+1}\theta-p_{i+1})+y(q_i\theta-p_i)\Longrightarrow$$

$$\Longrightarrow|k\theta-l|=|x(q_{i+1}\theta-p_{i+1})+y(q_i\theta-p_i)|=$$

$$=|x(q_{i+1}\theta-p_{i+1})|+|y(q_i\theta-p_i)|>|y(q_i\theta-p_i)|=$$

$$=|y||q_i\theta-p_i|\geq|q_i\theta-p_i|$$

The last inequality above implies the following:

$$|k\theta-l|>|q_i\theta-p_i|$$

or

$$\|k\theta\|>\|q_i\theta\|.$$

## III DYNAMIC PARTITION. RETURN TIME

Let $T(x)=\{x+\theta\}$ and $\theta$ is any irrational number on the interval $(0;1)$. We can write $\theta$ as continued fraction:

$$\theta=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\dots}}}$$

Let $x_0$ is any real number on $(0;1)$ and $T(x_0)=x_1, T^2(x_0)=T(T(x_0))=x_2,\dots,T^n(x_0)=T(T^{n-1}(x_0))=x_n$.

Properties:

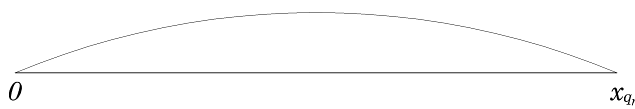
1. $x_{q_n}$ is located nearer to $x_0$ than any $x_i$ such that $q_n>i$. If $n$ is even, $x_{q_n}$ is located on the right side of $x_0$, else $x_{q_n}$ is located on the left side of $x_0$
2. Small distance between $x_0$ and $x_{q_n}$ is $\|q_n\theta\|$
3. The following equality holds:

$$\|x_m-x_n\|=\|x_k-x_l\|$$

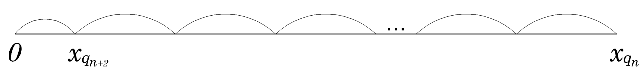
if $m-n=k-l$.

Now we consider "dynamic partition" of $T(x)=\{x+\theta\}$.

Consider the right neighborhood of $x_0$ and let there be some $x_{q_n}$:



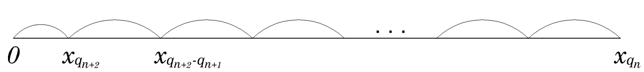

According to our condition, $x_{q_n}$ is located on the right side of $x_0$. By the 1st property, $x_{q_{n+2}}$ is also located on the right side of $x_0$. Main thing is that $x_{q_{n+2}}$ creates the interval $(x_0;x_{q_{n+2}})$ with a new length which is smaller than length of previous intervals:




Now define number of intervals which is formed between $x_{q_{n+2}}$ and $x_{q_n}$. $x_{q_{n+1}}$ also creates smaller interval than previous ones. Thus the intervals created by $x_{q_{n+1}}, x_{q_{n+1}+1},\dots,x_{q_{n+2}-1}$ are smallest until $x_{q_{n+2}}$ creates a new interval. Using the 2nd and 3rd property:

$$\|x_{q_{n+1}}-x_0\|=\|x_{q_{n+2}}-x_{q_{n+2}-q_{n+1}}\|.$$

Then the point on the right side of $x_{q_{n+2}}$ is $x_{q_{n+2}-q_{n+1}}$:

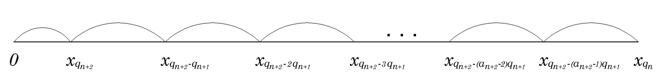



We can continue by this way:

$$\|x_{q_{n+1}}-x_0\|=\|x_{q_{n+2}}-x_{q_{n+2}-q_{n+1}}\|=\|x_{q_{n+2}-q_{n+1}}-x_{q_{n+2}-2q_{n+1}}\|$$

$$=\|x_{q_{n+2}-(a_{n+2}-1)q_{n+1}}-x_{q_{n+2}-a_{n+2}q_{n+1}}\|$$

But $q_{n+2}-a_{n+2}q_{n+1}=q_n$. There is $a_{n+2}$ point between $x_{q_{n+2}}$ and $x_{q_n}$. The graph is the following:




We can illustrate whole graph by that rule.

Now we formulate the theorem on "return time" for circle homeomorphisms with irrational rotation number in dynamical partitions. It is known that the first return time $R_E$ of an irrational rotation $T$ has at most three values if $E$ is an interval ([6], [7]). We present a proof using illustrations of dynamical partitions. Since $T$ is invariant translation, we may assume $E=[0,b)$.

**Theorem.** Let $T(x) = \{x+\theta\}$ and $b \in (0;\|\theta\|]$. Let $i \geq 0$ be an integer such that $\|q_i\theta\| < b \leq \|q_{i-1}\theta\|$ and $K$ an integer which satisfies

$$K = max\{k \geq 0\colon\ k\|q_i\theta\| + \|q_{i+1}\theta\| < b\}$$

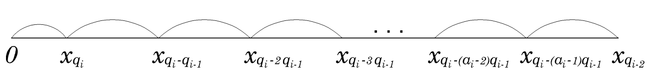
If $i$ is even, then

$$R_{[0;b]}(x) = \begin{cases} q_i, \\ \text{if } 0 \leq x < b - \|q_i\theta\|, \\ q_{i+1} - (K-1)q_i, \\ \text{if } b - \|q_i\theta\| \leq x < K\|q_i\theta\| + \|q_{i+1}\theta\|, \\ q_{i+1} - Kq_i, \\ \text{if } K\|q_i\theta\| + \|q_{i+1}\theta\| \leq x \leq b. \end{cases}$$

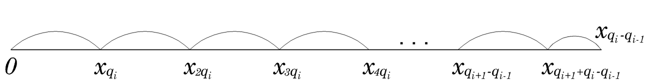
If $i$ is odd, then

$$R_{[0;b]}(x) = \begin{cases} q_{i+1} - Kq_i, \\ \text{if } 0 \leq x < b - K\|q_i\theta\| - \|q_{i+1}\theta\|, \\ q_{i+1} - (K-1)q_i, \\ \text{if } b - K\|q_i\theta\| - \|q_{i+1}\theta\| \leq x < \|q_i\theta\|, \\ q_i, \\ \text{if } \|q_i\theta\| \leq x \leq b. \end{cases}$$

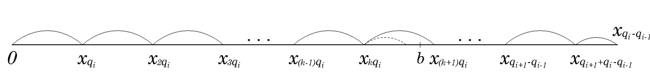
**Proof** with illustrations.
Let $x_0 = 0$ and $i$ be even. Let's consider the interval $[0;x_{q_{i-2}})$. We know that $x_{q_i}$ is nearer to $x_0$ than $x_{q_{i-2}}$ and $x_{q_i-q_{i-1}}$ is the nearest point to $x_{q_i}$ and the length of the interval $(x_{q_i};x_{q_i-q_{i-1}})$ is $\|q_{i-1}\theta\|$. Also, we have $a_i$ intervals between $x_{q_i}$ and $x_{q_{i-2}}$ that their length is also $\|q_{i-1}\theta\|$:




Let's consider the interval $[0;x_{q_i-q_{i-1}})$ only. The point $x_{q_{i+1}+q_i-q_{i-1}}$ takes the place that nearer to $x_{q_i-q_{i-1}}$. So, there will be $a_{i+1}$ intervals between $x_{q_i}$ and $x_{q_{i+1}+q_i-q_{i-1}}$ that their length is $\|q_i\theta\|$ which is the same as distance between $x_0$ and $x_{q_i}$:

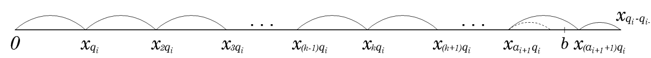



We know that smallest interval here is $(x_{q_{i+1}+q_i-q_{i-1}};x_{q_i-q_{i-1}})$ and its length is $\|q_{i+1}\theta\|$. Let's take the point $b$ that is located on $(0;x_{q_i-q_{i-1}})$:




Important note is that $b$ is not on $(x_{q_i-q_{i-1}} - \|q_{i+1}\theta\|;x_{q_i-q_{i-1}})$ because in this case $K$ will be $a_{i+1}$. If $K = a_{i+1}$, then:

$$a_{i+1}\|q_i\theta\| + \|q_{i+1}\| < b \Longrightarrow \|q_{i-1}\theta\| < b$$

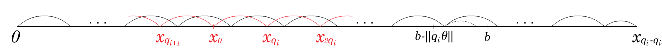



since $a_{i+1}\|q_i\theta\| + \|q_{i+1}\theta\| = \|q_{i-1}\theta\|$. But we have the condition

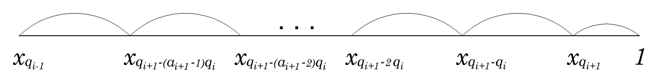
$$\|q_i\theta\| < b \leq \|q_{i-1}\theta\|.$$

In this case, we should say that $q_{i+1} + q_i - q_{i-1} = (a_{i+1}+1)q_i$. Also, $b$ cannot be located on $[0;x_{q_i})$, because of the condition above. Then $0 \leq K \leq a_{i+1} - 1$.

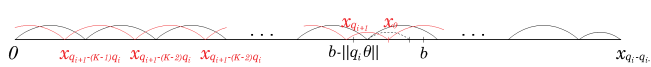
Now consider case when $x_0 \in [0;b]$. Let's consider the interval $[0;b - \|q_i\theta\|)$ first. If $x_0$ is on this interval, $x_{q_i}$ is the 1st point located on $[0;b]$:



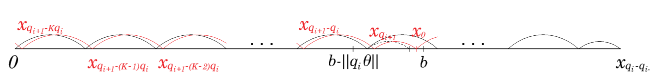

Let's illustrate the graph of $(x_{q_{i-1}};1]$ for $x_0 = 0$:



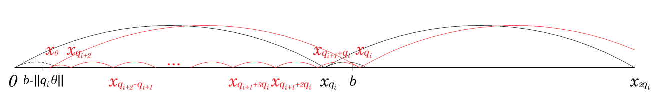

Let $x_0 \in [b - \|q_i\theta\|; K\|q_i\theta\| + \|q_{i+1}\theta\|)$. In this case, the 1st point located on $[0;b]$ is $x_{q_{i+1}-(K-1)q_i}$. We use the previous graph to illustrate it:




Let $x_0 \in [K\|q_i\theta\| + \|q_{i+1}\theta\|;b]$. In this case, the 1st point located on $[0;b]$ is $x_{q_{i+1}-Kq_i}$:




We state that the formula is true for $K = 0$. Let $K = 0$. $b$ will be on the interval $(x_{q_i};x_{q_i} + \|q_{i+1}\theta\|)$. If $x_0 \in [0;b - \|q_i\theta\|)$ or $x_0 \in [K\|q_i\theta\| + \|q_{i+1}\theta\|;b]$, it's easy to see that $R_{[0;b]} = q_i$ or $R_{[0;b]} = q_i - q_{i-1}$. We should proof that $R_{[0;b]} = q_{i+1} + q_i$, if $x_0 \in [b - \|q_i\theta\|;\ K\|q_i\theta\| + \|q_{i+1}\theta\|)$. Let's draw the graph and fill it with other points:



From the graph, you can see that $x_{q_{i+1}+q_i}$ is the 1st point located on

$$[b-\|q_i\theta\|;K\|q_i\theta\|+\|q_{i+1}\theta\|).$$

The theorem is proved for even i.

Let $i$ be odd. Let's move the main graph by 1 to the left side. Then we have the interval $(x_{q_{i-1}}-1;0]$ and let $-b$ be located on that interval but $b\neq 0$ and $\|q_i\theta\|<b\leq\|q_{i-1}\theta\|$. For this condition, we have following return time:

$$R_{[-b;0]}(x)=\begin{cases} q_{i+1}-Kq_i, \\ \text{if } -b\leq x<-K\|q_i\theta\|-\|q_{i+1}\theta\|, \\ q_{i+1}-(K-1)q_i, \\ \text{if } -K\|q_i\theta\|-\|q_{i+1}\theta\|\leq x<-b+\|q_i\theta\|, \\ q_i, \\ \text{if } -b+\|q_i\theta\|\leq x\leq 0. \end{cases}$$

If we shift the interval by $b$ to the right side, then we have:

$$R_{[0;b]}(x)=\begin{cases} q_{i+1}-Kq_i, \\ \text{if } 0\leq x<b-K\|q_i\theta\|-\|q_{i+1}\theta\|, \\ q_{i+1}-(K-1)q_i, \\ \text{if } b-K\|q_i\theta\|-\|q_{i+1}\theta\|\leq x<\|q_i\theta\|, \\ q_i, \\ \text{if } \|q_i\theta\|\leq x\leq b. \end{cases}$$

The proof has completed.

Now we compute return time of circle homeomorphisms using above theorem for exact irrational rotation number.

Let $\theta=\sqrt{2}-1$. Its continued fraction form is the following:

$$\theta=[2,2,\ldots,2,\ldots]=\frac{1}{2+\frac{1}{2+\frac{1}{2+\ldots}}}$$

Let's calculate return time for it:
Let $\|q_i\theta\|<b\leq\|q_{i-1}\theta\|$. Consider the interval $[0;b]$:
We know that

$$0\leq K\leq a_{i+1}-1$$

but have $a_k=2$ for any $k\in[1;\infty)$. So $a_{i+1}=1$. Thus, $K$ can be 0 or 1.
Let $K=0$:

In this case, $b$ cannot be greater than $\|q_{i+1}\theta\|+\|q_i\theta\|$ since $K$ will be 1. Then $b\in(\|q_i\theta\|;\|q_{i+1}\theta\|+\|q_i\theta\|]$.
If $i$ is even:

$$R_{[0;b]}(x)=\begin{cases} q_i, & 0\leq x<b-\|q_i\theta\|, \\ q_{i+1}+q_i, & b-\|q_i\theta\|\leq x<\|q_{i+1}\theta\|, \\ q_{i+1}, & \|q_{i+1}\theta\|\leq x\leq b. \end{cases}$$

If $i$ is odd:

$$R_{[0;b]}(x)=\begin{cases} q_{i+1}, & 0\leq x<b-\|q_{i+1}\theta\|, \\ q_{i+1}+q_i, & b-\|q_{i+1}\theta\|\leq x<\|q_i\theta\|, \\ q_i, & \|q_i\theta\|\leq x\leq b. \end{cases}$$

Let $K=1$. Then $b\in(\|q_{i+1}\theta\|+\|q_i\theta\|;\|q_{i-1}\theta\|]$.
If $i$ is even:

$$R_{[0;b]}(x)=\begin{cases} q_i, & 0\leq x<b-\|q_i\theta\|, \\ q_{i+1}, & b-\|q_i\theta\|\leq x<\|q_i\theta\|+\|q_{i+1}\theta\|, \\ q_{i+1}-q_i, & \|q_i\theta\|+\|q_{i+1}\theta\|\leq x\leq b. \end{cases}$$

If $i$ is odd:

$$R_{[0;b]}(x)=\begin{cases} q_{i+1}-q_i, & 0\leq x<b-\|q_i\theta\|-\|q_{i+1}\theta\|, \\ q_{i+1}, & b-\|q_i\theta\|-\|q_{i+1}\theta\|\leq x<\|q_i\theta\| \\ q_i, & \|q_i\theta\|\leq x\leq b. \end{cases}$$

## REFERENCES


[1] A. Khinchin. *Continued fractions*. 1964.

[2] Seo B.K. Kim, D.H. The waiting time for irrational rotations. *Nonlinearity 16 (5)*, pages 1861–1868, 2003.

[3] M. Kac. On the notion of recurrence in discrete stochastic processes. *Bull. Am. Math. Soc., 53*, pages 1002–10, 1947.

[4] Karimov J. Dzhalilov, A. The entrance times for circle maps with a break. *Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences: Vol. 3 : Iss. 2*, page Article 10, 2020.

[5] S.H. Yang. *Continued fractions and Pell's equation*. 2008.

[6] Berthe V. Alessandri, P. Three distance theorems and combinatorics on words. *Enseign. Math., 44*, pages 103–32, 1998.

[7] N. Slater. Gaps and steps for the sequence $n\theta(mod 1)$. *Proc. Camb. Phil. Soc., 63*, pages 1115–23, 1967.