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NUMERICAL SOLUTION OF THE NON-STATIONARY PROBLEM OF CHOOSING THE OPTIMAL PLACEMENT OF HEAT SOURCES IN A PARALLELEPIPED

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Abstract—In this paper, we study the problem of ensuring the temperature inside the field within the specified limits by choosing the optimal location of the heat sources in the parallelepiped. In this case, the optimal placement of heat sources on the area should be such that the total power of the consumed heat sources is minimal, so that the temperature is within the specified limits. By approximating the original problem, we obtain a difference equation. The construction of implicit difference schemes for the heat equation is given. From the difference equation it is reduced to a system of linear algebraic equations. The problem was solved using the M-method. In the parallelepiped, a new approach is proposed based on the numerical solution of the non-stationary problem of the optimal choice of the location of heat sources. Algorithms and software were developed for the numerical solution of the problem. A brief description of the software is provided. The results of a computational experiment are visualized.

Keywords— non-stationary problems, density, optimal choice, heat sources, implicit schemes, finite-dimensional approximation.

I INTRODUCTION

Many applied problems of modern natural science, in particular, the control of heat propagation in an environment, the mathematical model of which is the partial differential equation, lead to the choice of the location of heat sources in order to minimize the energy-dissipated amount of heat. In the process of mathematical modeling of systems related to resource-saving engineering technologies, the problem of optimal allocation of resources in heated rooms arises. The variety of optimization criteria allows setting a number of tasks. In fact, there are a number of problems here, which differ in the formulation of the problem and methods of solution. The problem of optimal placement of heat sources in heated rooms has always been relevant in construction, metallurgy, greenhouse design and various other areas of technology and technology. The problem of optimal placement of heat sources in heated rooms is also included in the general problem of this kind of applied problems. In this paper, we consider the problem of ensuring the temperature inside the region within the specified limits due to the optimal placement of heat sources of minimum power.

One of the distinctive features of this work is that a non-stationary problem is considered, i.e. the change in temperature depends not only on spatial variables, but also on time. It should be noted that in this case, a separate problem of optimization of a linear functional at each layer in time is considered.

One of the types of objects that are widespread in various fields of human activity are heat sources at the border, providing heat in a state of non-stationary heat balance with the environment. It is clear that the temperature inside the body depends on the temperature of the heating medium located at the border of the region. In a typical formulation, the problem of the optimal choice of the power of the heating medium is that the temperature field generated by them inside the body is in the given corridor. Similar problems arise in the organization of heating residential and industrial premises, greenhouses and, if necessary, to maintain a given temperature regime in homogeneous and inhomogeneous solids [1]. They allow a number of settings that are not equivalent due to differences in optimization criteria. Here we consider the problem of finding the density of heat sources of minimum power, which provides a given temperature regime in a certain body under the conditions of its non-stationary heat balance with the environment. Possible formulations and ways of solving the stationary problem are discussed in [2]. In the paper [3], a solution to the problem of optimal placement of sources in inhomogeneous media is proposed, in which scalar stationary fields are described by elliptic equations. The algorithms for solving the problem are based on effective methods of evaluating the values of the functional for a set of possible locations of sources, which makes it possible to choose the optimal option by implementing the branch and bound method in each specific case. In the work [4], the problems of optimal heating of a room based on the Pontryagin maximum principle are considered. The methodology for calculating the optimal control of transient modes during heating of the room is presented. The work [5] is devoted to the explicit formulation of the mathematical problem of optimization of heat supply in terms of its energy efficiency and the search for its solutions. In the work [6], a differential-difference problem of control of the diffusion process is studied, an analogue of the maximum principle is obtained, which makes it possible to determine such moments of switching on and off the maximum power of the source at which an admissible level of its concentration is established inside the parallelepiped at the observed concentration level of this substance at the boundary of the parallelepiped. [7] proposes a different solution concept.

This concept, in contrast to the classical one, allows points of discontinuity of the derivative with respect to the coordinate and, in contrast to the generalized solution, allows one to determine the derivative at the end points. Here, the inverse boundary value problem of thermal conductivity is posed and solved, provided that the thermal conductivity coefficient is piecewise constant. The problem was investigated using the Fourier series in eigenfunctions for an equation with a discontinuous coefficient. To solve the inverse problem, the Fourier transform was used, which allows the inverse problem to be reduced to an operator equation, which was solved by the residual method. In the works [8, 9], a method and an algorithm for solving the non-stationary problem of the optimal choice of the density of heat sources on simple geometric regions is developed so that the temperature inside the region under consideration is within the specified limits. In this case, the heat sources must provide a given temperature regime of the minimum total power and temperature in a given corridor filled with a homogeneous or inhomogeneous medium. In these works, to solve the problem, finite-dimensional approximations of the original problem are constructed in the form of a linear programming problem and the results of numerical experiments are presented. From a mathematical point of view, this problem belongs to the optimal control problems [10] for elliptic boundary value problems. The existence of a solution and general properties of similar problems for quadratic objective functionals, as well as approximate methods for their solution, have been studied by a number of authors [11, 12, 13]. Our problem can also be attributed to inverse heat conduction problems, methods of approximate solution of which are considered in [14]. In the paper [15], the third boundary value problem of parabolic type is considered. The distribution of heat in the body under consideration is controlled by a function that is located on the boundary of the body, the problem is solved, in the event of a conflict, about the possibility of transferring the initial position of the body to the desired state. Here we are talking mainly about the quadratic objective functional. In this work, the target functional is linear. Difficulties in solving the problem are generated by the inequalities included in the formulation that describe a given temperature regime. In addition, the absence of coercivity in a linear functional leads to difficulties in establishing the existence of a solution to the problem. We modify the problem formulation and construct a finite-dimensional approximation of this problem in the form of a sequence of linear programming problems. In this regard, we can assume that the solution of a finite-dimensional problem with a sufficiently large number is an approximate solution to the original problem. When solving this problem numerically, a number of difficulties arise, which have not been considered practically until now. An exact solution to this problem may not exist. In this work, the problem statement is refined and the so-called quasi-solution is introduced, which is quite acceptable from the applied point of view. Here, a method of finite-dimensional approximation of the problem is proposed, on the basis of which the main algorithms are developed, and a technique for the approximate finding of a quasi-solution is created.

In this paper, we consider the problem of finding the distribution of the density of heat sources, which provides a given temperature regime at the minimum total power of these sources. A method and an algorithm for solving non-stationary problems with the optimal choice of the density of heat sources on a parallelepiped in such a way that the temperature is within the specified limits are proposed. A software application has been created for carrying out computational experiments using this algorithm.

II Statement of the problem and its finite-dimensional approximation

Let the domain $D=\{a\leq x\leq b,\ c\leq y\leq d,\ p\leq z\leq q,\ 0\leq t\leq T\}$ be required to define a function $f(x,y,z,t)\geq 0$ that, for each $t\in[0,T]$, provides a minimum to the linear functional

$$J\{f\}=\int_a^b\int_c^d\int_p^q f(x,y,z,t)dzdydx\rightarrow\min, \qquad (1)$$

under the following conditions:

$$
\begin{gathered}
\frac{\partial u}{\partial t}=\chi\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+f(x,y,z,t),\\
x\in(a,b),\ y\in(c,d),\ z\in(p,q),\ t\in(0,T],\\
u(x,y,z,0)=u_0(x,y,z),\\
u(a,y,z,t)=\mu_1(y,z,t),\quad u(b,y,z,t)=\mu_2(y,z,t),\\
u(x,c,z,t)=\mu_3(x,z,t),\quad u(x,d,z,t)=\mu_4(x,z,t),\\
u(x,y,p,t)=\mu_5(x,y,t),\quad u(x,y,q,t)=\mu_6(x,y,t),\\
a\le x\le b,\ c\le y\le d,\ p\le z\le q,\ 0<t\le T.
\end{gathered}
\tag{2}
$$

$$
m(x,y,z,t)\le u(x,y,z,t)\le M(x,y,z,t),\ (x,y,z,t)\in D. \tag{3}
$$

Here $u=u(x,y,z,t)$ – is the temperature at the point (x,y,z) of the parallelepiped at the time t; χ – thermal diffusivity; $u_0(x,y,z)$, $\mu_1(y,z,t)$, $\mu_2(y,z,t)$, $\mu_3(x,z,t)$, $\mu_4(x,z,t)$, $\mu_5(x,y,t)$, $\mu_6(x,y,t)$, $m(x,y,z,t)$, $M(x,y,z,t)$ – are given continuous functions. Here $m(x,y,z,t)$, $M(x,y,z,t)$ – are functions of the minimum and maximum temperature profiles in the parallelepiped D, respectively. The density of heat sources is described by a square-integrable function $f(x,y,z,t)$ in the space $L_2(D)$. The solution to this boundary value problem can be obtained in an analytical form using the Fourier method [16].

Operator $Lu=\frac{\partial u}{\partial t}-\chi\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$ with initial and the boundary condition will be self-adjoint, positive definite in $L_2(D)$, which means that it has a bounded inverse operator $G=L^{-1}$. It can be used to reformulate the problem (1)-(3) as a problem on the minimum of the functional (1) under the following conditions on the density of sources:

$$
\begin{gathered}
m(x,y,z,t)\le (Gf)(x,y,z,t)\le M(x,y,z,t),\\
f(\cdot,\cdot,\cdot,\cdot)\in L_2(D),\quad f(x,y,z,t)\ge 0.
\end{gathered}
\tag{4}
$$

This problem in a mathematical formulation will be solved by the finite difference method on a uniform grid. To do this, divide the parallelepiped into N_1 in width, N_2 in length, N_3 in height into equal intervals, and construct, of course, a difference mesh.

Further, we replace the conditions (2) by their finite-difference analogs. In this case, we will use an implicit scheme.

Introduce in D a difference grid uniform in four variables $\overline{\omega}_{h_1h_2h_3\tau}=\overline{\omega}_{h_1}\times\overline{\omega}_{h_2}\times\overline{\omega}_{h_3}\times\overline{\omega}_{\tau}=\{(x_i,y_j,z_k,t_s)\colon\ x_i=ih_1,$ $y_j=jh_2,\ z_k=kh_3,\ t_s=s\tau,\ i=0,1,\ldots,N_1,\ \ j=0,1,\ldots,N_2,$ $k=0,1,\ldots,N_3,\ \ s=0,1,\ldots,N_4\}$ with steps $h_1=(b-a)/N_1$, $h_2=(d-c)/N_2$, $h_3=(q-p)/N_3$, $\tau=T/N_4$.

The implicit difference scheme for the (2) problem has the form [17]:

$$
\begin{cases}
\dfrac{u_{ijk}^{s+1}-u_{ijk}^{s}}{\tau}=\chi\Bigg(\dfrac{u_{i+1jk}^{s+1}-2u_{ijk}^{s+1}+u_{i-1jk}^{s+1}}{h_1^2}+\\
\dfrac{u_{ij+1k}^{s+1}-2u_{ijk}^{s+1}+u_{ij-1k}^{s+1}}{h_2^2}+\\
\dfrac{u_{ijk+1}^{s+1}-2u_{ijk}^{s+1}+u_{ijk-1}^{s+1}}{h_3^2}\Bigg)+f_{ijk}^{s+1},\ i=\overline{1,N_1-1},\\
j=\overline{1,N_2-1},\ k=\overline{1,N_3-1},\ s=\overline{0,N_4-1},\\
u_{ijk}^{0}=u_0(x_i,y_j,z_k),\\
u_{0jk}^{s+1}=\mu_1(y_j,z_k,t_{s+1}),\ u_{N_1jk}^{s+1}=\mu_2(y_j,z_k,t_{s+1}),\\
u_{i0k}^{s+1}=\mu_3(x_i,z_k,t_{s+1}),\ u_{iN_2k}^{s+1}=\mu_4(x_i,z_k,t_{s+1}),\\
u_{ij0}^{s+1}=\mu_5(x_i,y_j,t_{s+1}),\ u_{ijN_3}^{s+1}=\mu_6(x_i,y_j,t_{s+1}),\\
i=\overline{0,N_1},\ j=\overline{0,N_2},\ k=\overline{0,N_3},\ s=\overline{0,N_4-1}.
\end{cases}
\tag{5}
$$

Here $f_{ijk}^{s+1}=f(x_i,y_j,z_k,t_{s+1})+O(\tau+h_1^2+h_2^2+h_3^2)$.
Let us introduce the notation

$$
\begin{gathered}
XYZ=\left(\frac{1}{\tau}+\frac{2\chi}{h_1^2}+\frac{2\chi}{h_2^2}+\frac{2\chi}{h_3^2}\right),\\
X=-\frac{\chi}{h_1^2},\ Y=-\frac{\chi}{h_2^2},\ Z=-\frac{\chi}{h_3^2}.
\end{gathered}
$$

Consider the matrix

$$
A=\begin{bmatrix}
XYZ & Z & 0 & \ldots & 0 & Y & 0 & \ldots & 0 & X & 0 & \ldots & \ldots & 0\\
Z & XYZ & Z & 0 & \ldots & 0 & Y & 0 & \ldots & 0 & X & 0 & \ldots & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
0 & \ldots & 0 & X & 0 & \ldots & 0 & Y & 0 & \ldots & 0 & Z & XYZ & Z\\
0 & \ldots & \ldots & 0 & X & 0 & \ldots & 0 & Y & 0 & \ldots & 0 & Z & XYZ
\end{bmatrix}.
$$

We get

$$
G=A^{-1}.
$$

Let us construct a finite-dimensional approximation (1)-(5) as a linear programming problem. We divide the domain D into variables x,y,z,t, respectively, into N_1, N_2, N_3, N_4 of equal parts: $D=\bigcup_{s=1}^{N_4}\bigcup_{i=1}^{N_1}\bigcup_{j=1}^{N_2}\bigcup_{k=1}^{N_3}D_{ijk}^{s}$, where $D_{ijk}^{s}=\{(x,y,z,t),$ $x_{i-1}\le x\le x_i,\ y_{j-1}\le y\le y_j,\ z_{k-1}\le z\le z_k,\ t_{s-1}\le t\le t_s\}$, $i=\overline{1,N_1}$, $j=\overline{1,N_2}$, $k=\overline{1,N_3}$, $s=\overline{1,N_4}$. Denote by $S_{N_1N_2N_3}^{N_4}(D)\subset L_2(D)$ the subspace in which piecewise constant functions of the form $f(x,y,z,t) = f^s_{ijk}$, $(x,y,z,t) \in D^s_{ijk}$ $(i = \overline{1,N_1-1}$, $j = \overline{1,N_2-1}$, $k = \overline{1,N_3-1}$, $s = \overline{1,N_4})$. Introduce in $S^{N_4}_{N_1N_2N_3}(D)$ a basis consisting of the functions $e^s_{ijk}(x,y,z,t) = 1$, $(x,y,z,t) \in D^s_{ijk}$ and $e^s_{ijk}(x,y,z,t) = 0$, $(x,y,z,t) \notin D^s_{ijk}$. Then $f(x,y,z,t) \approx \sum_{s=1}^{N_4}\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} f^s_{ijk} e^s_{ijk}(x,y,z,t)$. Let $g_{rw} = (Ge^s_{ijk}, e^s_{ijk})$, $(m(x,y,z,t), e^s_{ijk}(x,y,z,t)) = m^s_{ijk}$, $(M(x,y,z,t), e^s_{ijk}(x,y,z,t)) = M^s_{ijk}$, $\tilde{f}^s_w = f^s_{ijk}$, $(r = w,\ w = (i-1)(N_2-1)(N_3-1)+(j-1)(N_3-1)+k)$, $N = (N_1-1)(N_2-1)(N_3-1)$, $r = \overline{1,N}$, $w = \overline{1,N}$, $i = \overline{1,N_1-1}$, $j = \overline{1,N_2-1}$, $k = \overline{1,N_3-1}$, $s = \overline{1,N_4}$, where $(\cdot,\cdot)$ – is a scalar product in $L_2(D)$. Substituting the expression for $f(x,y,z,t)$ in (1) and scalar multiplying the inequalities (4) by $e^s_{ijk}(x,y,z,t)$ in $L_2(D)$. As a result, we get the linear programming problem

$$
\begin{gathered}
J_s\{f\} = \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} (\mathrm{mes} D^s_{ijk}) f^s_{ijk} \to \min, \\
m^s_{ijk} \le \sum_{w=1}^{N} g_{rw}\tilde{f}^s_w \le M^s_{ijk}, \quad r = 1,2,...,N, \\
i = \overline{1,N_1-1},\ j = \overline{1,N_2-1},\ k = \overline{1,N_3-1},\ s = \overline{1,N_4}, \\
\tilde{f}^s_w \ge 0, \quad w = 1,2,...,N,\ s = 1,2,...,N_4.
\end{gathered} \tag{6}
$$

By solving the problem (6) numerically, we find the function $u^s_{ijk} = \sum_{w=1}^{N} g_{rw}\tilde{f}^s_w$, $(i = (r-1) \div (N_2-1)(N_3-1)+1)$ which is a solution to the boundary value problem (2) with $\tilde{f}^s_w$, where $\div$ is the integer division character. In this case, the (6) problem is solved by the simplex method [18].

III Description of algorithms and results of numerical experiments

For an approximate solution of the problem (1)-(6), software has been developed in the *C#* language. It allows you to represent all the necessary input data: constants, coefficients, mesh parameters, as well as temperature functions, initial and boundary conditions, in the form of scripts. Graphic modules have been developed to present the results.

The block diagram (Fig. 1) shows the general algorithm for solving the problem using the numerical method to calculate $J_{\min}$. For one and two-dimensional cases, a large number of computational experiments were carried out with different values of the input data.

Example 1. Find the optimal distribution density of sources on a parallelepiped. The cube $(0 \le x,y,z \le 1)$ with $\chi = 0.5$ m^2/s is used as the computational domain. The initial and boundary conditions are determined by the functions: $u_0(x,y,z) = 2+x^2+y^2+z^2$ m/s, $\mu_1(y,z,t) = 2+y^2+z^2+t^2$ m/s, $\mu_2(y,z,t) = 3+y^2+z^2+t^2$ m/s, $\mu_3(x,z,t) = 2+x^2+z^2+t^2$ m/s, $\mu_4(x,z,t) = 3+x^2+z^2+t^2$ m/s, $\mu_5(x,y,t) = 2+x^2+y^2+t^2$ m/s, $\mu_6(x,y,t) = 3+x^2+y^2+t^2$ m/s. The bounding temperature curves are given by the functions $m(x,y,z,t) = 1+x^2+y^2+z^2+t^2$ K, $M(x,y,z,t) = 4+x^2+y^2+z^2+t^2$ K and the end time is $T = 1$. Computational grid with the number of sources $(N_1-1)\times(N_2-1)\times(N_3-1)\times N_4 = 6\times6\times6\times7$. In fig. 2 presents the results of the numerical solution of the problem (6). The minimum for the numerical solution of the value of the functional is $J_{\min} = 17.31$ K·m/s. The results are presented with the minimum (blue borders), maximum (red borders) and approximate (green) temperature values. To illustrate the effectiveness of the developed method, Fig. 3 in the form of a histogram, the optimal distribution of sources is shown.

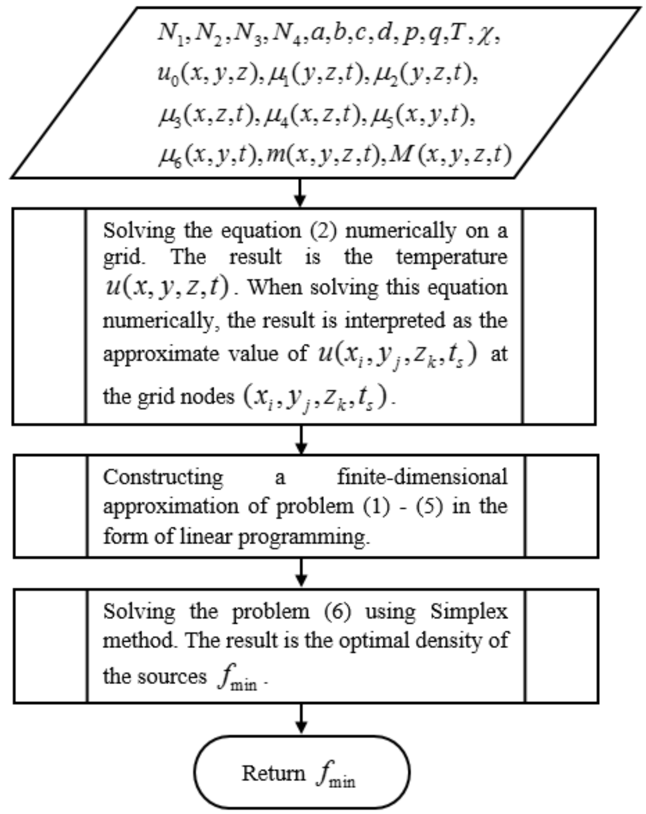

Fig. 1: Flowchart of the general algorithm for solving the problem

IV Conclusion

As you know, the construction of the Green's function for problems in partial derivatives, in fact, means finding a solution in an explicit form. When applying numerical methods, the values of the Green's function are represented in the form of a matrix, which is inverse to a matrix composed of the coefficients of a system of linear algebraic equations. Thus, it is possible to indicate the values of the required function at

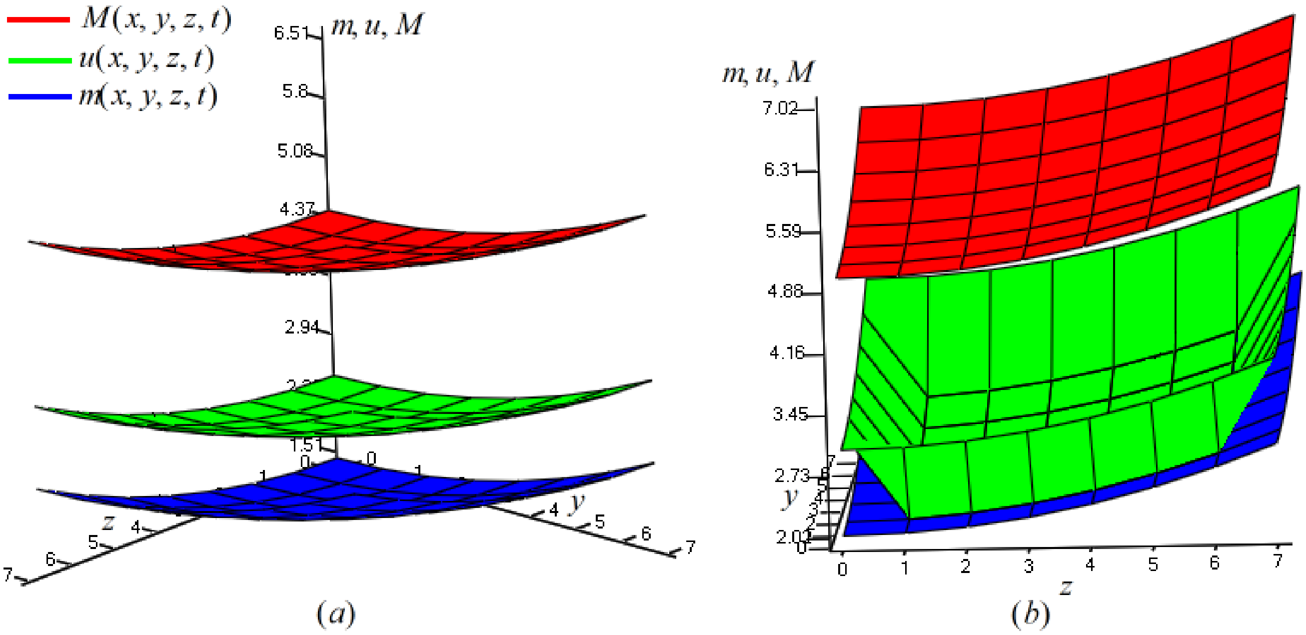

Fig. 2: Graph of the solution of the problem (6) at different times: for $x = 0.5$, $t = 0(a)$, for $x = 0.5$, $t = T(b)$. Solution to the problem lies in a given range is shown, i.e. the solution satisfies inequality (3). The value of $m(x,t)$ is practically equal to the minimum temperature can be seen. This means that the functional $J\{f\}$ reaches a minimum

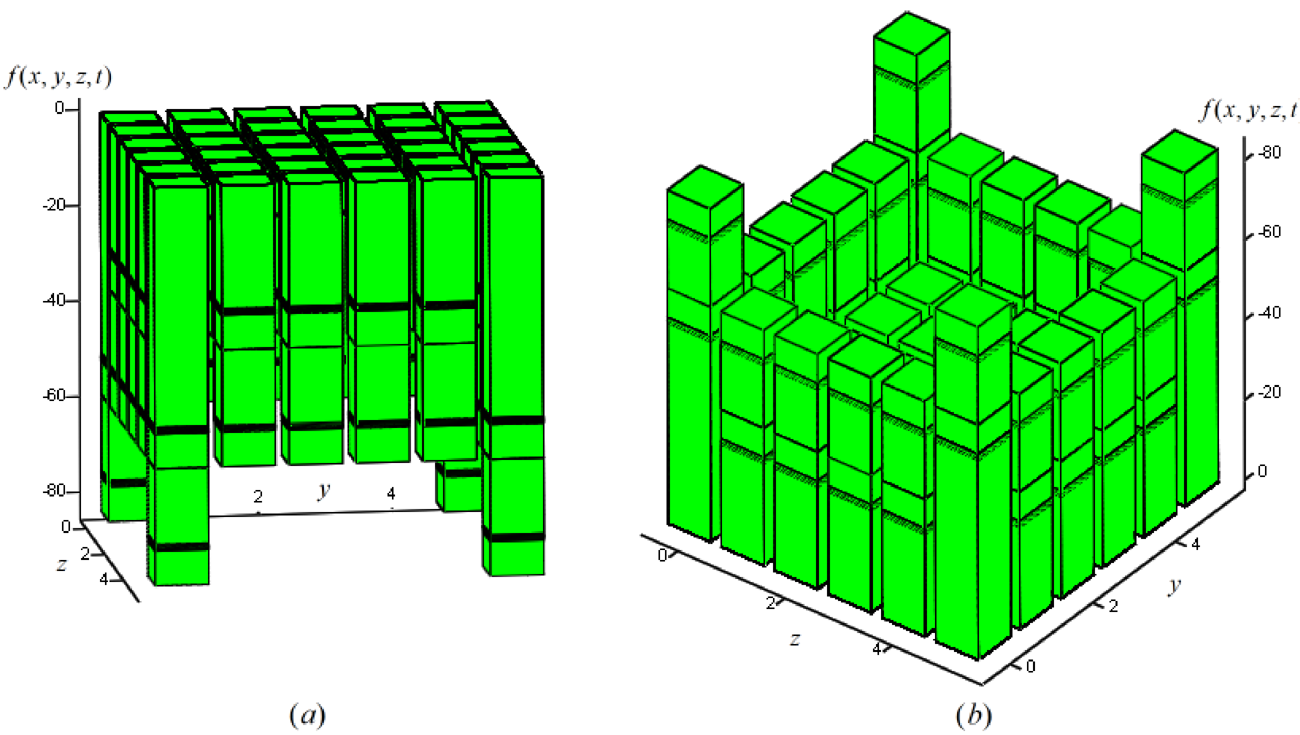

Fig. 3: Distribution of the optimal density of heat sources $f(x,y,z,t)$: in the usual (a) and inverted form (b). The power of the optimally placed heat sources is displayed as histogramme graph. All power sources are highlighted in green color. In general, high-power sources are located mainly on the border of the region

the nodal points of the partition. After substituting these values on the conditions-limitations of body temperature, taking into account the finding of the extremum of the functional, a linear programming problem is obtained, for the solution of which the standard M-method algorithm is used. Methods and algorithms for solving the non-stationary problem of ensuring the temperature inside the region within the specified limits by optimal placement of heat sources in the parallelepiped are proposed. The results of this computational experiment show that the functional has reached a minimum.

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