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On Hitting Times of Circle Maps with Generalized Dynamical Partitions

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Abstract—In present work we study the hitting times for circle homeomorphisms with one break point and universal renormalization properties. Consider the set $X(\rho)$ of all orientation preserving circle homeomorphisms $T \in C^{2+\varepsilon}(S^1 \setminus \{x_b\}),\ \varepsilon > 0$, with one break point x_b and irrational rotation number $\rho_T = \frac{-k+\sqrt{k^2+4}}{2}$, $k \geq 1$. For each $n \geq 1$ we define $c_n := c_n(c)$ such that $\mu([x_b, c_n]) = c \cdot \mu([x_b, T^{q_n}(x_b)])$, where q_n are first return times of T. Denote by $E_{n,c}(x)$ first hitting times of x to interval $[x_b, c_n]$. Consider the rescaled first hitting time $\overline{E}_{n,c} := \frac{1}{q_{n+1}} E_{n,c}(x)$. We study convergence in law of random variables $\overline{E}_{n,c}(x)$. We show that the limit distribution is singular w.r.t. Lebesgue measure.

Keywords—circle homeomorphisms, break point, rotation number,invariant measure, renormalization transformation, return time, hitting time

I Introduction

One of the important problems of ergodic theory is to study the behaviour of hitting times. Let $(X, \mathbb{B}, \mu)$ be a probability measure space and $T : X \to X$ be $\mu-$ invariant transformation. Fix a point $z \in X$ and consider the measurable subset $A \subset X,\ \mu(A) > 0$. Define the **hitting time** $E_A : X \to \mathbb{N}$ by

$$E_A(x) = inf\{i \geq 1 : T^i \in A\},$$

and we set $E_A(x) = \infty$, if $T^i(x) \notin A$ for all $i \in \mathbb{Z}^1$. The restriction of $E_A(x)$ to A is called the **first return time** of A.

The problem consists of finding conditions under which the hitting time, after rescaling by some suitable constant depending on A, converges in law, when $\mu(A)$ tends to zero. Since the expectation of the first hitting time is of the order $1/\mu(A)$, it is natural to rescaled the hitting time by this factor. Limit laws of hitting times have been obtained in various contexts such as: hyperbolic automorphisms of the torus and Markov chains [1], Axiom A diffeomorphisms and shifts of finite type with a Hölder potential, piecewise expanding maps of the circle and critical circle maps [2].

Circle homeomorphism, particularly theory of hitting time is important not only for the natural sciences, but also for applications in economics, information theory, biology, for the study of various heart diseases [3], in blood tests, etc.

The piecewise smooth circle homeomorphisms with break points studied by many authors (see for instance [4], [5], [6], [7], [8], [9], [10]).

Let T be an orientation preserving homeomorphism of the circle $S^1 = \mathbb{R}^1/\mathbb{Z}^1 \simeq [0,1)$ with irrational rotation number $\rho = \rho(T)$. Let $\mu = \mu_T$ be the unique invariant probability measure of T. Fix a point $z \in S^1$ and consider the interval $V_\varepsilon(z) = [z, z+\varepsilon] \subset S^1$. Consider the first hitting time to the interval $V_\varepsilon(z)$ by

$$E_\varepsilon(t) = inf\{i \geq 1 : T^i \in V_\varepsilon(z)\}.$$

Next define rescaled hitting time by $\overline{E}_\varepsilon(t) = \mu(V_\varepsilon(z))E_\varepsilon(t)$ We are interested in the converges of the distribution function of the random variable $\overline{E}_\varepsilon(t)$ i.e. in the convergence of the distribution function

$$F_\varepsilon(t) = \mu\left(x \in S^1 : \overline{E}_\varepsilon(x) \leq t\right),\ \forall t \in \mathbb{R}^1,$$

as $\varepsilon \to 0$, for every t belonging to the continuity points of the limit function. The case when $\varepsilon \to 0$, for every t belonging to the continuity points of the limit function.

Coelho and de Faria in [11], investigated the problem of convergence of random variables $\overline{E}_\varepsilon(t)$ for linear irrational rotations $T_\rho(x) = x + \rho\ mod 1$. It is known that for linear irrational rotation T_ρ unique invariant measure is Lebesgue measure ℓ. If $\varepsilon = \varepsilon_n$ is chosen such that $V_{\varepsilon_n}(z)$ corresponds to a sequence of renormalisation intervals for f_ρ as done in [11], it is proved in [11] that for Lebesgue almost every rotation number ρ, the rescaled hitting times $X_\varepsilon(t) = \mu(V_\varepsilon(z))E_\varepsilon(t)$ do not converge in law as ε tends to zero, and all possible limit laws under a subsequence of ε_n are obtained. Notice that if $F_{\varepsilon_n}(t)$ converges for some ε_n converging to zero every, the limit probability distribution $F(t)$ either step function with two discontinuity points, or uniform distribution on interval $[0,1]$. Let q_n, $n \geq 1$ be the first return times for T_ρ (see for details the next section). Fix $c \in (0,1]$. For every $n \geq 1$ uniquely define the points $c_n(\rho)$ by relation:

$$|[x_0, c_n(\rho)]| = c \cdot |[x_0, T_\rho^{q_n}(x_0)]|,$$

where $|\cdot|$ denotes the length of the in interval. We denote by $\Delta_{n,c}$ the interval $[x_0, c_n(\rho))$. Consider the hitting time $E_{n,c}$ to the interval $\Delta_{n,c}$. Define rescaled hitting time by

$$\overline{E}_{n,c}(x) := \mu(\Delta_{n,c}) E_{n,c}(x).$$

Denote by $F_{n,c}(t)$ the distribution function of $\overline{E}_{n,c}(x)$. It is proved that for any irrational number ρ, the rescaled hitting times $\overline{E}_\varepsilon(x)$ do not converge in law as ε tends to zero. In this paper we investigate the rescaled hitting times for circle homeomorphisms with a single break point. Let $T \in C^{2+\varepsilon}(S^1 \setminus x_0)$, $\varepsilon > 0$ be circle homeomorphism with a single break point x_0 and with irrational rotation number ρ_T i.e. $\rho_T = \frac{-k+\sqrt{k^2+4}}{2}$, $k \geq 1$. Denote by φ a conjugation between T and T_ρ, i.e. $\varphi \circ T = T_\rho \circ \varphi$.

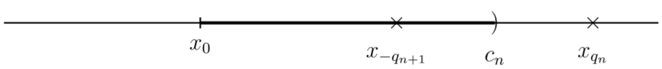

Figure 1.

On the circle S^1 there are two natural probability measures: Lebesgue measure ℓ and $T-$invariant measure $\mu := \mu_T$. Now we consider two distribution functions $F_{n,c}(t)$ and $\Phi_{n,c}(t)$ of rescaled hitting time $\overline{E}(x)$ with respect to measures μ and ℓ, respectively. The distribution function $F_{n,c}(t)$ of T coincide with distribution function of linear rotation T_ρ. Therefore all statements of Coelho and de Faria's work [11] are true for $T-$invariant measure μ. The invariant measure μ is singular with respect to Lebesgue [6]. Notice that for sufficiently piecewise smooth circle homeomorphisms with finite number break points and irrational rotation number the map T is ergodic w.r.t. Lebesgue measure also.

By definition

$$\Phi_n(t) = \ell\left(x \in S^1 : \overline{E}(x) \leq t\right), \ \forall t \in \mathbb{R}^1.$$

A.Dzhalilov in [2] studied the limit behaviour of distribution function with respect to Lebesgue measure for critical circle maps with irrational rotation number and for renormalized neighborhood of critical point. We study the limit behavior of distribution function with respect to Lebesgue measure for circle homeomorphism with single break point and with irrational rotation number.

The main results of our work are the following theorems.

Theorem 1. Let $c \in (0,1]$. Consider a circle homeomorphism $T \in C^{2+\varepsilon}(S^1 \setminus \{x_0\})$ with a single break point x_0 and with irrational rotation number with following expansion to continued fractions: $\rho = [k,k,\dots,k,\dots] = \frac{-k+\sqrt{k^2+4}}{2}$, $k \geq 1$. Let $\left\{\Phi_{n,c}(t)\right\}_{n=1}^{\infty}$ be the sequence of distribution functions with respect to Lebesgue measure on circle, corresponding to the first rescaled hitting time $\overline{E}_{n,c}(x)$ to interval $\Delta_{n,c}$. Then

1) For all $t \in R^1$ there exist the finite limit

$$\lim_{n\to\infty} \Phi_{n,c}(t) = \Phi_c(t),$$

and where $\Phi_c(t) = 0$, if $t \leq 0$, and $\Phi_c(t) = 1$, if $t > 1$;

2) $\Phi_c(t)$ is a strictly increasing on $[0,1]$ and continuous distribution function on R^1.

Theorem 2. Let $c \in (0,1]$. Consider a circle homeomorphism $T \in C^{2+\varepsilon}(S^1 \setminus \{x_0\})$ with a single break point x_0 and with irrational rotation number with following expansion to continued fractions: $\rho = [k,k,\dots,k,\dots] = \frac{-k+\sqrt{k^2+4}}{2}$, $k \geq 1$. Let $\left\{\Phi_{n,c}(t)\right\}_{n=1}^{\infty}$ be the sequence of distribution functions with respect to Lebesgue measure on circle, corresponding to the first rescaled hitting time $\overline{E}_{n,c}(x)$ to interval $\Delta_{n,c}$. Then $\Phi_c(t)$ is a singular function on the interval [0, 1], i.e. $\frac{d\Phi_c(t)}{dt} = 0$ for almost all (according to Lebesgue measure) $x \in [0, 1]$.

II Generalized dynamical partitions of the circle

Consider the circle homeomorphism T, with one break point b and irrational rotation number $\rho = \rho_T = [a_1, a_2, ..., a_n, ...]$. We let $\frac{p_n}{q_n}$ denote n th appropriate fraction of ρ_T, $p_n/q_n = [a_1, a_2, ..., a_n]$. The numbers q_n, $n \geq 1$ are called first return times Poincare for the map T. The numbers p_n and q_n satisfy recursion relations [12]:

$$p_{n+1} = a_{n+1} p_n + p_{n-1}, \ n \geq 1, \ p_0 = 0, \ p_1 = 1,$$

$$q_{n+1} = a_{n+1} q_n + q_{n-1}, \ n \geq 1, \ q_0 = a_1, \ q_1 = 1.$$

Via the orbit $\mathscr{O}_T = \{x_n = T^n(0), \ n \geq 0\}$ break point $x_0 = 0$ we define the sequence of dynamical partitions of a circle. We denote by $\Delta_0^{(n)}(x_0)$ the joining the points x_0 and $x_{q_n} = T^{q_n}(x_0)$.

Suppose $\Delta_i^{(n)} := T^i \Delta_0^{(n)}(x_0)$. It is well known, that the system of intervals

$$\mathbb{P}_n(x_0) = \{\Delta_0^{(n-1)}(x_0), \Delta_1^{(n-1)}(x_0), \dots, \Delta_{q_n-1}^{(n-1)}(x_0)\}$$

$$\cup \{\Delta_0^{(n)}(x_0), \Delta_1^{(n)}(x_0), \dots, \Delta_{q_{n-1}-1}^{(n)}(x_0)\}$$

is partition of the circle. Any two segments $\mathbb{P}_n(x_0)$ of partition can intersect only at end points. Partition $\mathbb{P}_n(x_0)$ called

$n-$th **dynamical partition**. We note, that $\mathbb{P}_1(x_0) < \mathbb{P}_2(x_0) < \ldots, \mathbb{P}_{n-1}(x_0) < \mathbb{P}_n(x_0), \ldots$. Under the transition from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$ all segments of rank n are preserved, and each of the segments with rank $(n-1)$ are decomposed into two segments:

$$\Delta_i^{(n-1)} = \Delta_i^{(n+1)} \cup \Delta_{i+q_n}^{(n)}.$$

Define the first return time function $R_{n,c} : \Delta_{n,c} \to \mathbb{N}$:

$$R_{n,c}(x) = \min\{j \geq 1 : T^j x \in \Delta_{n,c}\}.$$

From properties of dynamical partitions [7] we have

$$R_{n,c}(x) = \begin{cases} q_{n+1} \ , \ x \in [x_{-q_{n+1}}, c_n] \\ q_{n+2} \ , \ x \in [x_0, T^{-q_{n+2}} c_n] \\ q_{n+3} \ , \ x \in [T^{-q_{n+2}} c_n, x_{-q_{n+1}}] \end{cases} \quad (1)$$

We introduce the following notation:

$$A_0^{(n)} = [x_0, T^{-q_{n+2}} c_n], \ \ B_0^{(n)} = [x_{-q_{n+1}}, c_n]$$

$$C_0^{(n)} = [T^{-q_{n+2}} c_n, x_{-q_{n+1}}].$$

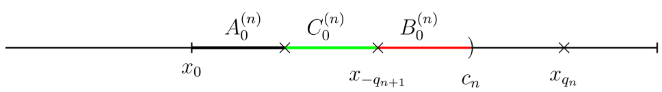

Figure 2.

Theorem 3. Consider the following segments $A_i^{(n)}$, $0 \leq i < q_{n+2}$, $B_j^{(n)}$, $0 \leq j < q_{n+1}$, $C_k^{(n)}$, $0 \leq k < q_{n+3}$, where $A_i^{(n)} := T^i(A_0^{(n)})$, $B_j^{(n)} := T^j(B_0^{(n)})$ and $C_k^{(n)} := T^k(C_0^{(n)})$. The following statements are hold:

1. $A_i^{(n)}$, $B_j^{(n)}$ and $C_k^{(n)}$ pairwise does not intersect (except for the end points);
2. $\left(\bigcup\limits_{i=0}^{q_{n+2}-1} A_i^{(n)} \right) \cup \left(\bigcup\limits_{j=0}^{q_{n+1}-1} B_j^{(n)} \right) \cup \left(\bigcup\limits_{k=0}^{q_{n+3}-1} C_k^{(n)} \right) = S^1$;
3.

$$\Delta_l^{(n)} = A_l^{(n)} \bigcup C_l^{(n)} \bigcup B_l^{(n)} \bigcup C_{l+q_{n+2}}^{(n)}, \ 0 \leq l < q_{n+1};$$

and

$$\Delta_s^{(n+1)} = A_{s+q_{n+1}}^{(n)} \bigcup C_{s+q_{n+1}}^{(n)}, \ 0 \leq s < q_n.$$

We prove this theorem by using main properties of dynamical partitions (see [12]). So first two statements easily follow from third statement and we show illustration of this statement.

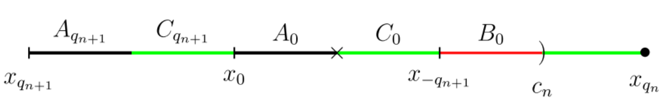

Figure 3.

III THE HITTING TIMES

Using (1) and Theorem 3 we obtain that $E_{n,c}(x)$ get values from 1 to q_{n+3}. Now normalize $E_{n,c}(x)$, i.e. we divide it by the largest value, and we denote by $\overline{E}_{n,c}(x)$, i.e.

$$\overline{E}_{n,c}(x) = \frac{1}{q_{n+3}} E_{n,c}(x).$$

Obviously, that function $\overline{E}_{n,c}(x)$ will be a random variable taking value in $[0,1]$. We denote by $\Phi_{n,c}(t)$ distribution function of $\overline{E}_{n,c}(x)$.

Now we formulate the next theorem.

Theorem 4. The distribution function of the normalized hitting function $\overline{E}_{n,c}(x)$ has the following form.

$$\Phi_{n,c}(t) = \begin{cases} 0, \text{ if } t \leq 0 \\[2ex] \sum\limits_{i=q_{n+1}-m}^{q_{n+1}-1} |A_i^{(n)}| + \sum\limits_{j=q_{n+2}-m}^{q_{n+2}-1} |B_j^{(n)}| + \sum\limits_{k=q_{n+3}-m}^{q_{n+3}-1} |C_k^{(n)}|, \\ \text{if } mq_{n+3}^{-1} \leq t \leq (m+1)q_{n+3}^{-1}, \ 1 \leq m \leq q_{n+1} \\[2ex] \sum\limits_{i=0}^{q_{n+1}-1} |A_i^{(n)}| + \sum\limits_{j=q_{n+2}-m}^{q_{n+2}-1} |B_j^{(n)}| + \sum\limits_{k=q_{n+3}-m}^{q_{n+3}-1} |C_k^{(n)}|, \\ \text{if } mq_{n+3}^{-1} \leq t \leq (m+1)q_{n+3}^{-1}, \ q_{n+1} \leq m \leq q_{n+2} \\[2ex] \sum\limits_{i=0}^{q_{n+1}-1} |A_i^{(n)}| + \sum\limits_{j=0}^{q_{n+2}-1} |B_j^{(n)}| + \sum\limits_{k=q_{n+3}-m}^{q_{n+3}-1} |C_k^{(n)}|, \\ \text{if } mq_{n+3}^{-1} \leq t \leq (m+1)q_{n+3}^{-1}, \ q_{n+2} \leq m \leq q_{n+3} \\[2ex] 1, \text{ if } t \geq 1 \end{cases}$$

Theorem 5. For all $n \geq 1$ the following relation is hold:

$$\Phi_{n,c}(t) = 1 - \Psi_{n,c}(1-t), \ t \in R^1,$$

where the distribution function $\Psi_{n,c}(t)$:

$$\Psi_{n,c}(t) = \begin{cases} 0, \text{ if } t \leq 0 \\ \sum\limits_{k=0}^{l} |C_k^{(n)}|, \text{ if } 1 \leq l < q_{n+1} \\ \sum\limits_{i=0}^{l-q_{n+1}} |A_i^{(n)}| + \sum\limits_{k=0}^{l} |C_k^{(n)}|, \text{ if } q_{n+1} \leq l < q_{n+2} \\ \sum\limits_{j=0}^{l-q_{n+2}} |B_j^{(n)}| + \sum\limits_{i=0}^{l-q_{n+1}-1} |A_i^{(n)}| + \sum\limits_{k=0}^{l} |C_k^{(n)}|, \text{ if } q_{n+2} \leq l < q_{n+3} \\ 1, \text{ if } t \geq 1 \end{cases}$$

where $l = q_{n+3} - m - 1$.

Proof. With using Theorem 4 and structure of dynamical partitions we write $\Phi_{n,c}(t)$ in useful form. Let $mq_{n+3}^{-1} \leq t \leq (m+1)q_{n+3}^{-1}$, $q_{n+2} \leq m \leq q_{n+3}$. We have

$$\Phi_{n,c}(t) = \sum_{i=q_{n+1}-m}^{q_{n+1}-1} |A_i^{(n)}| + \sum_{j=q_{n+2}-m}^{q_{n+2}-1} |B_j^{(n)}| + \sum_{k=q_{n+3}-m}^{q_{n+3}-1} |C_k^{(n)}|$$

Hence

$$\Psi_{n,c}(1-t)=1-\Phi_{n,c}(t)=1-\Bigg\{\sum_{i=q_{n+1}-m}^{q_{n+1}-1}|A_i^{(n)}|+$$

$$+\sum_{j=q_{n+2}-m}^{q_{n+2}-1}|B_j^{(n)}|+\sum_{k=q_{n+3}-m}^{q_{n+3}-1}|C_k^{(n)}|\Bigg\}=\sum_{k=0}^{q_{n+3}-m-1}|C_k^{(n)}|.$$

On the other hand

$$mq_{n+3}^{-1}\le 1-t\le (m+1)q_{n+3}^{-1},$$

$$1-(m+1)q_{n+3}^{-1}\le t\le 1-mq_{n+3}^{-1}.$$

From notation $m=q_{n+3}-l-1$

$$1-(q_{n+3}-l-1+1)q_{n+3}^{-1}\le t\le 1-(q_{n+3}-l-1)q_{n+3}^{-1};$$

$$lq_{n+3}^{-1}\le t\le (l+1)q_{n+3}^{-1},\ 1\le l<q_{n+1}.$$

From last inequality we get

$$\Psi_{n,c}(t)=\sum_{k=0}^{l}|C_k^{(n)}|,\ \text{ if } 1\le l<q_{n+1}.$$

If $mq_{n+3}^{-1}\le t\le (m+1)q_{n+3}^{-1}$ and $q_{n+1}\le m\le q_{n+2}$,

$$\Psi_{n,c}(1-t)=1-\Phi_{n,c}(t)=$$

$$=1-\Bigg\{\sum_{i=0}^{q_{n+1}-1}|A_i^{(n)}|+\sum_{j=q_{n+2}-m}^{q_{n+2}-1}|B_j^{(n)}|+\sum_{k=q_{n+3}-m}^{q_{n+3}-1}|C_k^{(n)}|\Bigg\}=$$

$$=\sum_{j=0}^{q_{n+2}-m}|B_j^{(n)}|+\sum_{k=0}^{q_{n+3}-m}|C_k^{(n)}|.$$

If one take

$$mq_{n+3}^{-1}\le 1-t\le (m+1)q_{n+3}^{-1},$$

$$1-(m+1)q_{n+3}^{-1}\le t\le 1-mq_{n+3}^{-1}.$$

Hence

$$lq_{n+3}^{-1}\le t\le (l+1)q_{n+3}^{-1},$$

$$\Psi_{n,c}(t)=\sum_{j=0}^{l-q_{n+1}}|B_j^{(n)}|+\sum_{k=0}^{l}|C_k^{(n)}|,\ \text{ if } q_{n+1}\le l<q_{n+2}.$$

If $mq_{n+3}^{-1}\le t\le (m+1)q_{n+3}^{-1},\ 1\le m\le q_{n+1}$,

$$lq_{n+3}^{-1}\le t\le (l+1)q_{n+3}^{-1}.$$

Hence,

$$\Psi_{n,c}(t)=\sum_{i=0}^{l-q_{n+2}-1}|A_i^{(n)}|+\sum_{j=0}^{l-q_{n+2}}|B_j^{(n)}|+\sum_{k=0}^{l}|C_k^{(n)}|,$$

if $q_{n+2}\le l<q_{n+3}$. Theorem 5 is completely proved.

REFERENCES

[1] Pitskel B. Poisson limit law for markov chains. *Ergodic Theory Dynamical Systems, V. 11*, pages 501–513, 1991.

[2] Dzhalilov A. A. Limiting laws for entrance times of critical mappings of a circle. *Theoret. and Math. Phys., 138:2*, pages 190–207, 2004.

[3] J. Belair D. Scagliotti D. Gordon M. Courtemanche, L. Glass. A circle map in a human heart. *Physica D: Nonlinear Phenomena, Volume 40, Issue 3*, pages 299–310, 1989.

[4] Vul E. B. Khanin K. M. Circle homeomorphisms with weak discontinuities. *Advances in Sov. Math, 3*, page 57–98, 1991.

[5] Safarov U. Dzhalilov A., Mayer D. Piecewise-smooth circle homeomorphisms with several break points. *Izvestiya RAN. Ser. Mat., V. 76, No. 1*, pages 95–113, 2012.

[6] Khanin K.M. Dzhalilov A.A. On an invariant measure for homeomorphisms of a circle with a point of break. *Funct. Anal. Its Appl., No. 32*, pages 153–161, 1998.

[7] Karimov J.J. On continuity of limit distribution function for rescaled hitting times. *Uzbek Mathematical Journal, No. 4*, pages 78–88, 2019.

[8] Dzhalilov A.A. Karimov J.J. The entrance times for circle maps with a break the entrance times for circle maps with a break. *Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences: Vol. 3: Iss. 2*, page Article 10, 2020.

[9] Dzhalilov A. A. Karimov J. J. The thermodynamic formalism and exponents of singularity of invariant measure of circle maps with a single break. *Bulletin of Udmurt University. Mathematics, Mechanics, Computer Science*, pages 343–366, 2020.

[10] Dzhalilov A. A. Karimov J. J. On exponents of probability invariant measure of circle maps with a break point. *AIP Conference Proceedings 2781*, pages 020059–1–020059–6, 2023.

[11] De Faria E. Coelho Z. Limit laws of entrance times for homeomorphisms. *Israel J. Math., 93*, pages 93–112, 1996.

[12] Sinai Ya.G. *Topics in Ergodic Theory*. 1994.