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First-order moving average processes associated by interval exchange maps

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Abstract– In present work we investigate the nonlinear first-order moving average processes associated by interval exchange maps h. Let random process $\mathbb{X} := \{X_n,\ n \geq 1\}$ defined by

$$X_{n+1}(h) := h(\xi_n) + \xi_{n+1}\ ,\ n \in \mathbb{Z},$$

where $\widehat{\xi} := \{\xi_n,\ n \geq 1\}$ is independent, identically uniformly distributed on interval $[0,1]$ random sequence. We investigate the random process $\mathbb{X}$ for stationarity and find their distribution function and autocovariance function.

Key words– moving average process, interval exchange map, strictly stationary process, covariance function.

I Introduction

A Time Series is a sequence of dates indexed by time. Every data is a discrete observation taken from an underlying process. Time series analysis looks at the methods used to create the models from the sampled data in order to study the continuous process. This paper will focus on the first-order moving average process (MA(1)). Notice that the model MA(1) is one of classical models in the theory of time series. It is very important and applied in different problems of practice (see for instance [6], [2],[5],[4]).

Let $a, b \in (0,1)$. Consider the interval exchange map $h : [0,1) \to [0,1)$ defined by (see [1]):

$$h(x) := \begin{cases} a + \frac{1-a}{b}x,\ 0 \leq x < b, \\ \frac{a}{1-b}(x-b),\ b \leq x < 1. \end{cases} \quad (1)$$

The graph of h is shown in Figure 1 on the next page. The map h has two break points $x = 0$ and $x = b$ with **jump ratios**

$$\sigma_f(0) := \frac{f'_-(0)}{f'_+(0)} = \frac{ab}{(1-a)(1-b)},$$

$$\sigma_f(b) := \frac{f'_-(b)}{f'_+(b)} = \frac{(1-a)(1-b)}{ab},$$

Let A^* denote the class of all such maps. Recall that x_0 is called a break point of a map f if $\frac{f'_-(x_0)}{f'_+(x_0)} = \sigma_f(x_0) \neq 1$ and $f_\pm(x_0) > 0$. It is obvious that $\sigma_f(0) \cdot \sigma_f(1) = 1$. Only in the case $a = b = \frac{1}{2}$ we have $\sigma_f(0) = \sigma_f(b) = 1$. Identifying the endpoints of the interval $[0,1)$ we get we get unit circle $S^1 = \mathbb{R}/\mathbb{Z}$. In this case by the map h uniquely can be defined orientation preserving circle homeomorphism.

It is easy to see that the inverse function h^{-1} is

$$h^{-1}(x) := \begin{cases} b + \frac{1-b}{a} \cdot x,\ 0 \leq x < a, \\ \frac{b}{1-a}(x-a),\ a \leq x < 1. \end{cases} \quad (2)$$

Definition 1.1. (see [7], [3]). Let $\mathbb{X} := (X_n), \in T \subset \mathbb{Z}$ be a stationary random process. The **autocovariance function (ACVF)** of $\{X_n\}$ at lag m is

$$\gamma_X(m) = Cov(X_{t+m}, X_t) = E\left[(X_{t+m} - E[X_{t+m}])(X_t - E[X_t])\right]$$

The **autocorrelation function (ACF)** of $\mathbb{X}$ at lag m is

$$\rho_X(m) = \frac{\gamma_X(m)}{\gamma_X(0)}.$$

Definition 1.2. The process $\mathbb{X} := (X_n), \in T \subset \mathbb{Z}$ is said to be **weakly stationary** if

1) $E(X_t)$ is independent of t,

and

2)$\gamma_X(t+h, t)$ is independent of t for each h.

Definition 1.3. The random process $\mathbb{X} := (X_n), \in T \subset \mathbb{Z}$ is said to be **strictly stationary** if $(X(t_1), ..., X(t_k))$ and $(X(t_1+m), ..., X(t_k+m))$ have the same joint distribution for all integers $t_1, ..., t_k \in T,\ k \geq 1$ and $m > 0$.

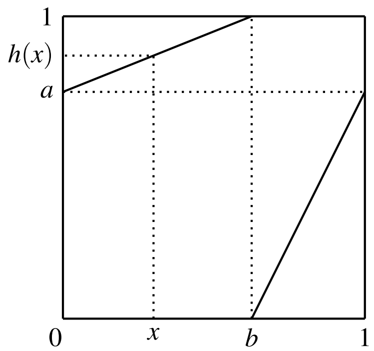

Figure 1

Let (Ω, F, P) be a probability space. Let $\{\xi_n, n \in \mathbb{Z}\}$ be a sequence of independent, identically distributed (i.i.d.) random variables with uniform distribution $[0,1]$ i.e. its **probability density function (pdf)** is

$$f_{\xi_0}(x) := \begin{cases} 1, \ x \in [0,1], \\ 0, \ otherwise. \end{cases}$$

Next we define the following sequence of random variables associated by map h :

$$X_{n+1}(h) := h(\xi_n) + \xi_{n+1} \ , \ n \in \mathbb{Z}. \tag{3}$$

The random prosess $\mathbb{X}(h) := \{X_{n+1}(h), \ n \in \mathbb{Z}\}$ is called **first-order moving average or MA(1) random process associated by map** h**.**

We investigate the sequence of random variables $\{h(\xi_n), \ n \geq 1\}$.

Theorem 1.1. If h is the interval map defined by (1) and random variable ξ is uniformly distributed on the interval $[0,1]$. Then the **cumulative distribution function (cdf)** of $h(\xi)$ has form

a) $$F_{h(\xi)}(t) = \begin{cases} 0, \ \text{if } t < 0, \\ \frac{1-b}{a}t, \ \text{if } 0 \leq t < a, \\ \frac{b}{1-a}t + \frac{1-a-b}{1-a}, \ \text{if } a \leq t \leq 1, \\ 1, \ \text{if } t > 1. \end{cases}$$

b) $E[h(\xi)] = \frac{1}{2}(a+b)$,
where $E[h(\xi)]$ is the expectation of $h(\xi)$.

We formulate the main result of our work.

Theorem 1.2. Let $\mathbb{X}(h) := \{X_n(h), \ n \in \mathbb{Z}\}$ be the MA(1) process defined by (3). Then

(1) $\mathbb{X}(h)$ is strictly stationary random process;

(2) The density function of each $X_n(h)$ has the following form

$$(3) \quad f_{\mathbb{X}}(t) := f_{X_n}(t) = \begin{cases} \frac{1-b}{a}t, \ \text{if } t \in [0,a), \\ \frac{b}{1-a}t + \frac{1-a-b}{1-a}, \ \text{if } t \in [a,1), \\ -\frac{1-b}{a}t + \frac{1+a-b}{a}, \ \text{if } t \in [1,1+a), \\ \frac{b}{1-a}(2-t), \ \text{if } t \in [1+a,2), \\ 0, \ \text{otherwise}; \end{cases}$$

(4) For every $n \in \mathbb{Z}$ the expectation of X_n can be written as

$$E[X_n] = \frac{1}{2}(1+a+b);$$

(5) The variance of X_n can be written as

$$Var(X_n) = \frac{1}{12}\left(a^2 - 2ab - 3b^2 + 4b + 1\right);$$

(6) The autocorrelation function (ACF) of $\mathbb{X}$:

$$\rho(1) = \frac{-2ab + a + 4b^2 - 3b}{a^2 - 2ab - 3b^2 + 4b + 1},$$

and

$$\rho(m) = 0 \quad \text{for} \ \ m \geq 2.$$

II PROOFS OF THE THEOREMS 1.1 AND 1.2

Proof of Theorem 1. By definition of random process $\mathbb{X}(h) := \{X_n(h), \ n \in \mathbb{Z}\}$ it is easy to see that it ia strictly stationary process.

Note that the map h is invertible. Using this and the distribution function of uniformly distributed random variable ξ_0 we find the distribution of $h(\xi_0)$:

$$F(t) := P\left\{\xi_0 < h^{-1}(t)\right\} = \begin{cases} 0, \ t < 0, \\ \frac{1-b}{a}t, \ 0 \leq t < a, \\ \frac{b}{1-a}t + \frac{1-a-b}{1-a}, \ a \leq t \leq 1, \\ 1, \ t > 1, \end{cases}$$

Consequently, the probability density function of $h(\xi_0)$ is

$$f_{h(\xi)}(t) = F'_{h(\xi)}(t) = \begin{cases} \frac{1-b}{a}, \ 0 \leq t < a, \\ \frac{b}{1-a}, \ a \leq t \leq 1, \\ 0, \ otherwise \end{cases}$$

Now we evaluate the expectation of $h(\xi_0)$:

$$
\begin{aligned}
E\left[h\left(\xi_n\right)\right] &= \int_{-\infty}^{\infty} h(x) f_{\xi_n}(x)\,dx \\
&= \int_0^b \left(a+\frac{1-a}{b}x\right)dx + \int_b^1 \frac{a}{1-b}(x-b)\,dx \\
&= \frac{ab^2}{2} + \frac{(1-a)b^2}{3} + \frac{a}{1-b}\left(\frac{1}{3}-\frac{b}{2}-\frac{b^3}{3}+\frac{b^3}{2}\right) \\
&= \frac{1}{2}(a+b)
\end{aligned}
$$

TheoremI is completely proved. □

Proof of Theorem 2.

Since the sequence of random variables $\{\xi_n, n\geq 0\}$ is independent and identically distributed, using the definition of $\{X_k, k\geq 1\}$ we can decide that it is identically distributed also.

The density function of X_1 random variable can be found by the following formula (see [8])

$$
f_{X_1}(t) = \int_{-\infty}^{\infty} f_{h(\xi_0)}(x) f_{\xi_1}(t-x)\,dx
$$

Because $f_{\xi_1}(x)$ and $f_{h(\xi_0)}(x)$ functions are zero outside$[0,1]$ interval ,we get following double inequality $\begin{cases} 0\leq x\leq 1 \\ t-1\leq x\leq t \end{cases}$

Then

1) if $0\leq t\leq a$ we have $0\leq x\leq t$ and

$$
f_{X_1}(t) = \int_0^t f_{h(\xi_0)}(x) f_{\xi_1}(t-x)\,dx = \int_0^t \frac{1-b}{a}dx = \frac{1-b}{a}t
$$

2) if $a\leq t\leq 1$ we have $0\leq x\leq t$ and

$$
\begin{aligned}
f_{X_1}(t) &= \int_0^t f_{h(\xi_0)}(x) f_{\xi_1}(t-x)\,dx \\
&= \int_0^a \frac{1-b}{a}dx + \int_a^t \frac{b}{1-a}dx \\
&= \frac{1-b}{a}a + \frac{b}{1-a}(t-a) = \frac{b}{1-a}t + \frac{1-a-b}{1-a}
\end{aligned}
$$

3) if $1\leq t\leq 1+a$ we have $t-1\leq x\leq 1$ and

$$
\begin{aligned}
f_{X_1}(t) &= \int_{t-1}^1 f_{h(\xi_0)}(x) f_{\xi_1}(t-x)\,dx \\
&= \int_{t-1}^a \frac{1-b}{a}dx + \int_a^1 \frac{b}{1-a}dx \\
&= \frac{1-b}{a}(a-t+1) + \frac{b}{1-a}(1-a) \\
&= -\frac{1-b}{a}t + \frac{1+a-b}{a}
\end{aligned}
$$

4) if $1+a\leq t\leq 2$ we have $t-1\leq x\leq 1$ and

$$
\begin{aligned}
f_{X_1}(t) &= \int_{t-1}^1 f_{h(\xi_0)}(x) f_{\xi_1}(t-x)\,dx = \int_{t-1}^1 \frac{b}{1-a}dx \\
&= \frac{b}{1-a}(2-t)
\end{aligned}
$$

We have

$$
f_{X_1}(t) = \begin{cases}
\frac{1-b}{a}t,\ t\in[0,a), \\
\frac{b}{1-a}t + \frac{1-a-b}{1-a},\ t\in[a,1), \\
-\frac{1-b}{a}t + \frac{1+a-b}{a},\ t\in[1,1+a), \\
\frac{b}{1-a}(2-t),\ t\in[1+a,2), \\
0, otherwise
\end{cases}
$$

We evaluate the expectation of X_1 :

$$
\begin{aligned}
E[X_1] &= \int_{-\infty}^{\infty} x\cdot f_{X_1}(x)\,dx \\
&= \int_0^a \frac{1-b}{a}x^2dx + \int_a^1 \left(\frac{b}{1-a}x^2 + \frac{1-a-b}{1-a}x\right)dx \\
&+ \int_1^{1+a} \left(\frac{b-1}{a}x^2 + \frac{1+a-b}{a}x\right)dx + \int_{1+a}^2 \frac{b}{1-a}\left(2x-x^2\right)dx \\
&= \frac{1-b}{3a}a^3 + \frac{b}{3(1-a)}\left(1-a^3\right) + \frac{1-a-b}{2(1-a)}\left(1-a^2\right) \\
&+ \frac{b-1}{3a}\left((1+a)^3-1\right) + \frac{1+a-b}{2a}\left((1+a)^2-1\right) \\
&+ \frac{b}{1-a}\left(2^2-\frac{2^3}{3}-(1+a)^2+\frac{(1+a)^3}{3}\right) \\
&= \frac{2a^2(1-b)}{6} + \frac{2a^2b-3a^2-ab-b+3}{6} \\
&+ \frac{2a^2b+a^2+3ab+3a}{6} + \frac{2b\left(2-a-a^2.\right)}{6} \\
&= \frac{1}{2}(1+a+b)
\end{aligned}
$$

We find the second moment of X_1 :

$$
\begin{aligned}
E\left[(X_1)^2\right] &= \int_{-\infty}^{\infty} x\cdot f_{X_1}(x)\,dx \\
&= \int_0^a \frac{1-b}{a}x^3dx + \int_a^1 \left(\frac{b}{1-a}x^3+\frac{1-a-b}{1-a}x^2\right)dx \\
&+ \int_1^{1+a}\left(\frac{b-1}{a}x^3+\frac{1+a-b}{a}x^2\right)dx \\
&+ \int_{1+a}^{2}\frac{b}{1-a}\left(2x^2-x^3\right)dx = \frac{1-b}{4a}a^4 \\
&+ \frac{b}{4(1-a)}\left(1-a^4\right)+\frac{1-a-b}{3(1-a)}\left(1-a^3\right) \\
&+ \frac{b-1}{4a}\left((1+a)^4-1\right)+\frac{1+a-b}{3a}\left((1+a)^3-1\right) \\
&+ \frac{b}{1-a}\left(\frac{2\cdot 2^3}{3}-\frac{2^4}{4}-\frac{2(1+a)^3}{3}+\frac{(1+a)^4}{4}\right) \\
&= \frac{3a^3(1-b)}{12}+\frac{3a^3b-4a^3-a^2b-ab-b+4}{12} \\
&+ \frac{3a^3b+a^3+8a^2b+4a^2+6ab+6a}{12} \\
&+ \frac{-3a^3b-7a^2b-ab+11b}{12} \\
&= \frac{1}{6}\left(2a^2+2ab+3a+5b+2\right)
\end{aligned}
$$

Using the expectation and the second moment we evaluate its variance (see [7], [3]):

$$
\begin{aligned}
Var[X_1] &= E\left[(X_1)^2\right]-(E[X_1])^2 \\
&= \frac{1}{6}\left(2a^2+2ab+3a+5b+2\right)-\left(\frac{1}{2}(1+a+b)\right)^2 \\
&= \frac{1}{12}\left(a^2-2ab-3b^2+4b+1\right)
\end{aligned}
$$

We are interested in finding the autocovarience between $X_1$and X_2. First we will find some quantities needed to find autocovarience.

1)

$$E[\xi_0] = \int_{-\infty}^{\infty} x\cdot f_{\xi_n}(x)\,dx = \int_0^1 1dx = \frac{1}{2}$$

2)

$$
\begin{aligned}
E[\xi_0\cdot h(\xi_0)] &= \int_{-\infty}^{\infty} x\cdot h(x)\,f_{\xi_0}(x)\,dx \\
&= \int_0^b\left(a+\frac{1-a}{b}x\right)xdx+\int_b^1\frac{a}{1-b}(x-b)\,xdx \\
&= \frac{ab^2+2b^2}{6}+\frac{2a-ab-ab^2}{6} \\
&= \frac{1}{6}\left(2b^2+2a-ab\right)
\end{aligned}
$$

3)

$$
\begin{aligned}
E[X_2X_1] &= E[(\xi_2+h(\xi_1))(\xi_1+h(\xi_0))] \\
&= E[\xi_2]\cdot E[\xi_1]+E[\xi_2]\cdot E[h(\xi_0)] \\
&+ E[\xi_1\cdot h(\xi_1)]+E[h(\xi_1)]\cdot E[h(\xi_0)] \\
&= \frac{1}{4}+\frac{a+b}{4}+\frac{2b^2+2a-ab}{6}+\left(\frac{a+b}{2}\right)^2 \\
&= \frac{1}{12}\left(3a^2+4ab+7a+7b^2+3b+3\right)
\end{aligned}
$$

and its the autocovariance function $h=1$

$$
\begin{aligned}
\gamma(1) &= Cov[X_2\cdot X_1] = E[(X_2-E[X_2])(X_1-E[X_1])] \\
&= E[X_2X_1-X_2E[X_1]-X_1E[X_2]+E[X_2]E[X_1]] \\
&= E[X_2X_1]-(E[X_1])^2 \\
&= \frac{1}{12}\left(3a^2+4ab+7a+7b^2+3b+3\right)-\left(\frac{1+a+b}{2}\right)^2 \\
&= \frac{1}{12}\left(-2ab+a+4b^2-3b\right)
\end{aligned}
$$

and its the autocorelation function(ACF) for $h=1$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{-2ab+a+4b^2-3b}{a^2-2ab-3b^2+4b+1}$$

□

We define a new random process on the unit circle:

$$Z_n = X_n \mod 1,\ n\geq 1.$$

The random process $\mathbb{Z} := \{Z_n\ n\geq 1\}$ is strictly stationary . We find the density function and the moments of $\mathbb{Z} := \{Z_n\ n\geq 1\}$.

Using the density function of X_1 we can write the density of Z_1 :

$$F_{Z_n}(z) = P(Z_1\leq z) = P(\{X_1\leq z\}\cup\{1<X_1\leq 1+z\})$$

1) In the case $0\leq z<a$, we have

$$
\begin{aligned}
F_{Z_1}(z) &= \int_0^z\frac{1-b}{a}tdt+\int_1^{1+z}\left(\frac{b-1}{a}t+\frac{1+a-b}{a}\right)dt \\
&= \frac{1-b}{a}\cdot\frac{z^2}{2}+\frac{b-1}{2a}\left((1+z)^2-1\right)-\frac{1+a-b}{a}z = z
\end{aligned}
$$

2) If $a\leq z<1$, we have

$$
\begin{aligned}
F_{Z_1}(z) &= \int_0^a \frac{1-b}{a} t\,dt + \int_a^z \left(\frac{b}{1-a}t + \frac{1-a-b}{1-a}\right)dt \\
&+ \int_1^{1+a} \left(\frac{b-1}{a}t + \frac{1+a-b}{a}\right)dt + \int_{1+a}^{1+z} \frac{b}{1-a}(2-t)\,dt \\
&= \frac{1-b}{a}\cdot\frac{a^2}{2} + \frac{b}{2(1-a)}\left(z^2-a^2\right) + \frac{1-a-b}{1-a}(z-a) \\
&+ \frac{b-1}{2a}\left((1+a)^2-1\right) - \frac{1+a-b}{a}(1+a-1) \\
&+ \frac{b}{1-a}\left(2(1+z)-2(1+a)\right) \\
&- \frac{b}{1-a}\left(\frac{(1+z)^2}{2} - \frac{(1+a)^2}{2}\right) = \frac{a-ab}{2} \\
&+ \frac{b}{1-a}\left(\frac{z^2}{2}-z\right) + z + \frac{2a^2-ab^2-2a+2ab}{2(1-a)} \\
&+ \frac{ab+a}{2} + \frac{b}{1-a}\left(-\frac{z^2}{2}+z\right) + \frac{ab^2-2ab}{2(1-a)} = z
\end{aligned}
$$

Finally we have

$$
F_{Z_1}(z) = \begin{cases} 0, z<0; \\ z, 0 \le z \le 1; \\ 1, z>1. \end{cases}
$$

It is obvious that Z_1 is uniform distribution on interval $[0,1]$. We evaluate its expectation and variance :

$$
1) E[Z_1] = \int_{-\infty}^{\infty} z \cdot f_{Z_1}(z)\,dz = \int_0^1 z\,dz = \frac{1}{2}
$$

$$
\begin{aligned}
Var[Z_1] &= \int_{-\infty}^{\infty} (z - E[Z_1])^2 f_{Z_1}(z)\,dz \\
&= \int_0^1 \left(z-\frac{1}{2}\right)^2 dz = \frac{1}{12}.
\end{aligned}
$$

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