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INVARIANT MEASURE OF CRITICAL CIRCLE HOMEOMORPHISMS WITH COUNTABLE NUMBER OF BREAKS

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Abstract– It is proved that invariant measures of P - homeomorphisms of a circle with countable many break points and with single critical point, and an irrational rotation number are singular with respect to the Lebesgue measure on the circle.

Key words– circle homeomorphism, rotation number, invariant measure, critical point, break point.

In the present paper, we study invariant measures of circle homeomorphisms with singularities, more precisely, critical circle homeomorphisms with an infinite number of breaks.

Consider an orientation-preserving circle homeomorphism f:

$$f(x) = F(x)(mod\,1),\ x \in S^1,$$

where $F(x)$ - continuous, strictly increasing function on R^1 satisfying condition

$$F(x+1) = F(x)+1, x \in R,$$

and it is called a **lift** of the homeomorphism f. Let F be the lift with initial condition $F(0) \in [0,1)$. We denote by ρ_f the rotation number of the homeomorphism f (see [1]), that is,

$$\rho_f = \lim_{n\to\infty} \frac{F^n(x)}{n},\ x \in R^1.$$

Henceforth, $F^n(x)$ denotes the nth iteration of the function F. The limit ρ_f belongs to the interval $[0,1)$ and does not depend on the point x . The number ρ_f is the most important numerical characteristic of the homeomorphism f. Namely, if the rotation number ρ_f is irrational, then the homeomorphism f has a unique probability invariant measure μ_f. Moreover, there exists a continuous, non-decreasing function $\varphi : S^1 \to S^1$ such that $\varphi \circ f = f_{\rho_f} \circ \varphi$, where $f_{\rho_f} = x+\rho_f(mod\,1)$ (see [1]). Note that the semi-conjugation φ and the invariant measure μ_f are connected by the relation $\varphi(x) = \mu_f([0,x]), x \in S^1$. Because of this relation, invariant measure μ_f is absolutely continuous with respect to the Lebesgue measure ℓ if and only if φ is given by an absolutely continuous function.

The fundamental results in the problem of smoothness of the conjugacy were obtained by V.I. Arnold, J. Moser, M. Herman, J.C. Yoccoz, Ya.G. Sinai and K.M. Khanin, Y. Katznelson and D. Ornstein.

We formulate the last two important results in this area.

Theorem 1. (Katznelson-Ornstein,[2]). *Let f be an orientation preserving C^1- circle diffeomorphism. If f' is absolutely continuous, $\frac{f''}{f'} \in L_p$ for some $p > 1$ and the rotation number $\rho = \rho_f$ is of bounded type, then the invariant measure μ_f is absolutely continuous with respect to Lebesgue measure.*

Theorem 2. (Khanin-Sinai,[3]). *Let f be a $C^{2+\varepsilon}$ circle diffeomorphism for some $\varepsilon > 0$, and let the rotation number $\rho = \rho_f$ be a Diophantine number with exponent $\delta \in (0,\varepsilon)$, i.e., there is a constant $c(\rho)$ such that*

$$|\rho - \frac{p}{q}| \geq \frac{c(\rho)}{q^{2+\delta}},\ for\ any\ p,q \in Q$$

Then the conjugating map φ belongs to $C^{1+\varepsilon-\delta}$.

One of the important class of circle homeomorphisms are homeomorphisms with break points, or the class of P - homeomorphisms.

Definition 1. *Let f be circle homeomorphisms with the lift F. If at the point $x_b \in S^1$ there exist one-sided positive derivatives $F'(x_b-0)$, $F'(x_b+0)$ and $F'(x_b-0) \neq F'(x_b+0)$, then $x = x_b$ is called break point of the homeomorphism f.*

The number $\sigma_f(x_b) = \frac{F'(x_b-0)}{F'(x_b+0)}$ is called **jump ratio** or **jump** of the homeomorphism f at the point $x = x_b$.

Definition 2. *An orientation-preserving circle homeomorphism f with the lift F is called P - homeomorphism, if F satisfies the following conditions:*

1) F is differentiable on S^1 except at a finite or countable number of break points;

2) there exist constants $0 < c_1 < c_2 < +\infty$ such that

$$c_1 < F'(x_b - 0), F'(x_b + 0) < c_2, \ \forall x_b \in BP(f),$$

$$c_1 < F'(x) < c_2, \ \forall x \in S^1 \setminus \{BP(f)\},$$

where $BP(f)$ - set of all break points of f;

3) $\ln F'$ has bounded variation in S^1, i.e. $v(F) = var_{S^1} \ln F' < \infty$.

The regularity properties of invariant probability measures of circle homeomorphisms with break points differ from the properties in the case of circle diffeomorphisms. The piecewise-linear (PL) orientation preserving circle homeomorphisms with two break points are the simplest examples of P-homeomorphisms. The invariant measures of PL homeomorphisms were studied first by Herman in [4].

Theorem 3. (Herman). *A PL circle homeomorphisms with two break points and irrational rotation number has an invariant measure absolutely continuous with respect to Lebesgue measure if and only if its break points belong to the same orbit.*

General (non PL) circle homeomorphisms with one break point have been studied by Dzhalilov and Khanin in [5]. The character of their results for such circle maps is quite different from the one for $C^{2+\varepsilon}$- diffeomorphisms. The main result of [5] is the following:

Theorem 4. *Let f be a circle homeomorphisms with a single break point x_b. If the rotation number ρ_f of f is irrational and $f \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$ for some $\varepsilon > 0$, then the $f-$ invariant probability measure μ_f is singular with respect to Lebesgue measure ℓ.*

The invariant measures of circle homeomorphisms with two break points of "general type", that is, which are not piecewise linear, were studied in [6], [7]. We state the main results of these papers.

Theorem 5. ([6]). *Suppose that a circle homeomorphism f with lift F satisfies the following conditions.*

1) The rotation number ρ_f is irrational of "bounded type", that is, the sequence of elements of the expansion of ρ_f into a continued fraction is bounded.

2) f has break points at two points b_1, b_2 of the circle that do not lie on the same trajectory.

3) The derivative $F'(x)$ exists on the set $S^1 \setminus \{b_1, b_2\}$ and satisfies Lipschitz conditions on every connected component of that set.

Then the $f-$invariant measure μ_f is singular with respect to Lebesgue measure ℓ.

Circle homeomorphisms with two break points but arbitrary irrational rotation number were studied in [7].

Theorem 6 ([7]). *Suppose that a circle homeomorphism f with lift F satisfies the following conditions.*

1) The rotation number ρ_f is irrational;

2) f has break points at points b_1, b_2 and the derivative $F'(x)$ is absolutely continuous on every connected component of the set $S^1 \setminus \{b_1, b_2\}$;

3) $F''(x) \in L_1(S^1, d\ell)$;

4) The product of the jumps at the break points is nontrivial, that is, $\sigma_1 \cdot \sigma_2 \neq 1$.

Then the $f-$ invariant probability measure μ_f is singular with respect to Lebesgue measure.

Now we formulate the main result of the paper of A.A. Dzhalilov, D. Mayer, U.A. Safarov [8].

Theorem 7. *Suppose that the lift $F(x)$ of circle homeomorphism f with irrational rotation number satisfies the following conditions:*

(1) *f has break points $b(1), b(2), ..., b(k) \in S^1$ and $F'(x)$ absolutely continuous function on each connected component of the set $S^1 \setminus \{b(i), i = \overline{1,k}\}$;*

(2) $\int_{S^1} |F''(x)| d\ell < \infty$;

(3) $\prod_{i=1}^{k} \sigma_i \neq 1$.

Then the f - invariant probability measure μ_f is singular with respect to Lebesgue measure ℓ on the circle S^1, i.e. there exists a set $A \subseteq S^1$ such that $\ell(A) = 1$ and $\mu_f(A) = 0$.

In the paper [9] author answered positively a question of whether it is possible for a circle diffeomorphisms with breaks to be smoothly conjugate to a rigid rotation in the case when its breaks are lying on pairwise distinct trajectories. An example constructed is a piecewise linear circle homeomorphisms that has four break points lying on distinct trajectories, and whose invariant measure is absolyutely continuous w.r.t. the Lebesgue measure.

Another important class of circle homeomorphisms with singularities is critical circle homeomorphisms.

Definition 3. *The point $x_{cr} \in S^1$ is called non-flat critical point of a homeomorphism f with order $d > 1$, if $f(x) = \phi(x)|\phi(x)|^{d-1} + f(x_{cr})$ for all x in the some $\delta-$neighborhood $U_\delta(x_{cr})$, where $\phi : U_\delta(x_{cr}) \to \phi(U_\delta(x_{cr}))$ is a C^3 diffeomorphism such that $\phi(x_{cr}) = 0$.*

An important one-parameter family of examples of critical circle maps are the Arnold's maps defined by

$$f_\theta(x) := x + \theta + \frac{1}{2\pi} \sin 2\pi x \, (mod 1), \, x \in S^1$$

For every $\theta \in R^1$ the map f_θ is a critical map with critical point 0 of cubic type.

Invariant measures of critical circle homeomorphisms were studied for the first time by Graczyk and Swiatek ([10]). They proved that if f is C^3 smooth circle homeomorphism with finitely many critical points of polynomial type and an irrational rotation number of bounded type, then the invariant measure of critical circle homeomorphisms is singular w.r.t. Lebesque measure on S^1. Recently de Faria and Guarino ([11]) proved the following

Theorem 8. *If $f : S^1 \to S^1$ is a C^3 multicritical circle map without periodic points, then f admits no $\sigma-$ finite invariant measure which is absolutely continuous with respect to Lebesgue measure.*

There arises then naturally the problem of regularity of the invariant measure of circle homeomorphisms of mixed types of singularities, that is, homeomorphisms with critical and break points. Regularity of invariant measure of circle homeomorphisms with finite number of mixed types of singularities was studied in [12].

The main result of [12] is the following

Theorem 9. *Suppose that circle homeomorphisms f has critical points $x_{cr}^{(k)}$, $k = \overline{1,m}$ with order $d_k > 1$ and its rotation number ρ_f is irrational. Let f be $P-$ homeomorphism in the set $S^1 \setminus \bigcup_{k=1}^{m} U_{\omega_k}(x_{cr}^{(k)})$ with finite number of break points. Then $f-$ invariant measure μ_f is singular w.r.t. Lebesgue measure.*

Now we formulate our main result.

Theorem 10. *Suppose that circle homeomorphisms f has only one critical point x_{cr} with order $d > 1$ and rotation number ρ_f is irrational. Let f be $P-$ homeomorphism in the set $S^1 \setminus U_\omega(x_{cr})$ with countable many break points. Then $f-$ invariant measure μ_f is singular w.r.t. Lebesgue measure.*

I CONCLUSION

The question of the absolute continuity and singularity of two probability measures is one of the important problems of modern probability theory. These results confirm that for critical circle homeomorphisms with countable many break points and with an irrational rotation number, it is proved that the invariant probability measure is singular with respect to the Lebesgue measure.

II REFERENCES

[1] A. Denjoy: Sur les courbes d´efinies par les ´equations diff´erentielles ‘a la surface du tore. J. Math. Pures Appl., 11, 333-375 (1932).

[2] Y. Katznelson and D. Ornstein: The absolute continuity of the conjugation of certain diffeomorphisms of the circle. Ergod. Theor. Dyn. Syst., 9, 681-690, (1989).

[3] K.M. Khanin and Ya.G. Sinai: Smoothness of conjugacies of diffeomorphisms of the circle with rotations. Russ. Math. Surv., 44, 69-99, (1989), translation of Usp. Mat. Nauk, 44, 57-82, (1989).

[4] M. Herman: Sur la conjugaison diff´erentiable des diff´eomorphismes du cercle ‘a des rotations. Inst. Hautes Etudes Sci. Publ. Math., 49, 225-234 (1979).

[5] A.A. Dzhalilov and K.M. Khanin: On invariant measure for homeomorphisms of a circle with a point of break., Funct. Anal. Appl., 32, (3) 153-161 (1998).

[6] A.A. Dzhalilov and I. Liousse: Circle homeomorphisms with two break points. Nonlinearity, 19, 1951-1968 (2006).

[7] A.A. Dzhalilov, I. Liousse and D. Mayer: Singular measures of piecewise smooth circle homeomorphisms with two break points. Discrete and continuous dynamical systems, 24, (2), 381-403 (2009).

[8] A.A. Dzhalilov, D. Mayer and U.A. Safarov: Piecwise-smooth circle homeomorphisms with several break points. Izvestiya RAN: Ser. Mat. 76:1, 95-113, translation of Izvestiya: Mathematics 76:1, 95-113, (2012).

[9] Teplinsky A.: A circle diffeomorphism with breaks that is absolutely continuously linearizable. Ergodic Theory and Dynamical Systems, 38, N1, 371–383, (2018).

[10] J. Graczyk, G. ´Swi¸atek.: Singular measure in circle dynamics. Commun. Math. Phys. 157, 213-230, (1993).

[11] P. Guarino, E. de Faria: There are no σ-finite absolutely continuous invariant measures for multicritical circle maps. Nonlinearity, 34, N10, 6727–351, (2021).

[12] U.A. Safarov: Invariant measure of circle maps with mixed type of singularities. Izvestiya vuzov, Mathematics, N7, 71–84, (2023).