Acta of Turin Polytechnic University in Tashkent, 2024, 30, 18-26
Published Online March 2024 in Acta TTPU (http://www.acta.polito.uz/)

The Neumann eigenvalue problem for the $p(x)$ - Laplacian as $p \to \infty$

Farhod Abdullayev
Department of Natural and Mathematical Sciences, Turin Polytechnic University in Tashkent
Email: f.abdullaev@polito.uz

Abstract– This paper is dedicated to the study of the behaviour of the second eigenvalues and the corresponding eigenfunctions for the $p(x)$-Laplacian subject to the Neumann boundary conditions in an open, bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. As $p \to \infty$ one can obtain uniform bounds for the sequence of second eigenvalues and the positive second eigenfunctions. In the latter case, the uniform limit is a viscosity solution to a problem involving the ∞-Laplacian subject to appropriate boundary conditions.

Key words– ∞-Laplacian, eigenvalue problems, Luxemburg norm, $p(x)$-Laplacian, variable exponent Lebesgue and Sobolev spaces, viscosity solutions.

I INTRODUCTION

During the last decades, a lot of studies have been dedicated to the understanding the partial differential equations with non-standard growth conditions in the framework of variable exponent spaces. Its applications arise in many areas such as in electrorheological fluids (see (27)), image processing (see (6)) and nonlinear elasticity (see (2), (30)). In particular, a lot of attention has been paid to the study of eigenvalue problems for the $p(x)$-Laplace operator

$$-\Delta_{p(x)} := -\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \Lambda_{p(\cdot)}|u|^{p(x)-2}u$$

in open bounded domains $\Omega \subset \mathbb{R}^N$, subject to Dirichlet ((15), (16)), Neumann ((17)), Robin ((7), (28)) and Steklov ((8)) boundary conditions.

A number of papers have been concentrated on the asymptotic analysis of solutions to partial differential equations involving the $p(x)$-Laplacian as $p(x) \to \infty$ (see (23), (18), (20), (21), (22), (25), (26)). For the case of Dirichlet and Robin boundary conditions, the asymptotic behavior of the first eigenvalue/eigenfunction pairs associated to $-\Delta_{p(x)}$ has been studied in (25) and (1), respectively. In this paper we study the asymptotic behavior, as $p \to \infty$, of the second eigenvalues and the corresponding eigenfunctions for the $p(x)$-Laplacian with Neumann boundary conditions:

$$\begin{cases} -\Delta_{p(x)}u = \Lambda|u|^{p(x)-2}u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\eta = \eta(x)$ stands for the outer unit normal to $\partial\Omega$ at $x \in \partial\Omega$.

To analyze the limiting behavior of this problem as $p \to \infty$ we replace $p = p(x)$ above by $p_n = p_n(x)$, where $\{p_n\} \subset C^1(\overline{\Omega})$ is a sequence of functions that satisfies the following conditions:

(i) $p_n \to \infty$ uniformly in Ω;

(ii) $\bigtriangledown ln p_n \to \xi \in C(\bar{\Omega}, \mathbb{R}^N)$ uniformly in Ω;

$p_n \to \infty, \bigtriangledown ln p_n \to \xi \in C(\bar{\Omega}, \mathbb{R}^N), and \frac{p_n}{n} \to q \in C(\bar{\Omega}, (0, +\infty))$ uniformly in Ω and aims to analyze what happens with the solutions of the problem at level n as $n \to \infty$. These conditions on the sequence p_n are typical in the literature (see, e.g. (22), (25), (26), or (20), (18) for the particular case $p_n(\cdot) = np(\cdot)$ - corresponding to $\xi = \nabla \ln p$ and $q = p$). We prove that after eventually extracting a subsequence, the (positive) second eigenfunctions converge uniformly in $\Omega \subset \mathbb{R}^N$ to a viscosity solution of the problem

$$\begin{cases} \min\left\{-\Delta_\infty u - |\nabla u|^2 \ln|\nabla u|\langle\xi, \nabla u\rangle, |\nabla u|^q - \Lambda_\infty|u|^q\right\} = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_∞ is the ∞-Laplace operator, $\Delta_\infty u := \sum_{i,j=1}^{N} u_{x_i}u_{x_j}u_{x_ix_j}$, Λ_∞ is the limit of the sequence of (suitably rescaled) second eigenvalues.

The paper is organized as follows. In Section 2 we give the definition and some basic properties of variable exponent Lebesgue and Sobolev spaces. Section 3 of the paper is devoted to the Neumann eigenvalue problem for $-\Delta_{p(x)}$ for the case where $p = p(x)$ is fixed. After stating the definition of a weak solution, we review some details concerning the Ljusternik-Schnirelman existence theory for this problem, and we show that continuous weak solutions are also solutions in the viscosity sense. Here, we adopt the definition of viscosity solutions for second-order elliptic equations with fully nonlinear boundary conditions introduced by Barles in (3). Finally, in Section 4 we state and prove the main result of the paper, Theorem **??**, regarding the convergence of the second eigenvalues and the corresponding positive eigenfunctions as $p(\cdot) \to \infty$.

II Preliminaries

In this section, we provide a brief introduction to variable exponent Lebesque and Sobolev spaces. For more details we refer to the books by Diening, Harjulehto, Hästö & M. Ružička (10), Musielak (24), and the papers by Edmunds, Lang & Nekvinda (11), Edmunds & Rákosník (12; 13), and Kovacik & Rákosník (19).

Let $\Omega \subset R^N$ be an open set with smooth boundary, and let $|\Omega|$ stand for the N-dimensional Lebesgue measure of Ω. Given any continuous function $p : \overline{\Omega} \to (1,\infty)$, let $p^- := \inf_{x\in\Omega} p(x)$ and $p^+ := \sup_{x\in\Omega} p(x)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

It is a Banach space when endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)} := \inf \left\{ \mu > 0 \ : \ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

For constant functions p the space $L^{p(\cdot)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, endowed with the standard norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{1/p}.$$

$L^{p(\cdot)}(\Omega)$ is separable and reflexive if $1 < p^- \le p^+ < +\infty$. If $0 < |\Omega| < \infty$ and if p_1, p_2 are variable exponents such that $p_1 \le p_2$ in Ω then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous, and its norm does not exceed $|\Omega| + 1$.

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. The following version of Hölder's inequality

$$\left| \int_\Omega uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)}, \tag{II.1}$$

for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ holds. The *modular* of the space $L^{p(\cdot)}(\Omega)$ is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) := \{ u \in L^{p(\cdot)}(\Omega) : \ |\nabla u| \in L^{p(\cdot)}(\Omega) \},$$

and it becomes a Banach space when endowed with one of the equivalent norms

$$\|u\|_{p(\cdot)} := |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)},$$

or

$$\|u\| := \inf \left\{ \mu > 0; \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \le 1 \right\},$$

where in the definition of $\|u\|_{p(\cdot)}$, $|\nabla u|_{p(\cdot)}$ stands for the Luxemburg norm of $|\nabla u|$. Under very mild assumptions on the function p, the space $W^{1,p(\cdot)}(\Omega)$ is also separable and reflexive. Another important fact that we will use in the sequel is that the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact and continuous if $p(x) \ge \alpha > N$, $\forall\, x \in \Omega$. The following extensions of the classical results for Lebesgue spaces are well-known (see, e.g., (10)).

Lemma 1 *Let $\{f_n\}$ be a sequence of measurable functions. If $f_n \to f$ and $|f_n(x)| \le g(x)$ a.e. $x \in \Omega$ for some $f : \Omega \to \mathbb{R}$ measurable and $g \in L^{p(\cdot)}(\Omega)$, then $f_n \to f$ in $L^{p(\cdot)}(\Omega)$.*

Lemma 2 *Let $\{u_n\} \subset L^{p(\cdot)}(\Omega)$ and $u \in L^{p(\cdot)}(\Omega)$. The following statements are equivalent:*

- *(i)* $\lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0$;
- *(ii)* $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$;
- *(iii)* $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u)$.

III The Neumann Eigenvalue Problem for the $p(x)$-Laplacian

Let Ω be an open bounded domain with smooth boundary, and consider the Neumann eigenvalue problem for the $p(x)$-Laplacian

$$\begin{cases} -\Delta_{p(x)} u = \Lambda |u|^{p(x)-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{III.1}$$

where $\eta = \eta(x)$ stands for the outer unit normal to $\partial\Omega$ at $x \in \partial\Omega$.

Definition 1. We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a weak solution for the Neumann eigenvalue problem (III.1) if there exists $\Lambda_{p(\cdot)} \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx = \Lambda_{p(\cdot)} \int_\Omega |u|^{p(x)-2} uv \quad dx, \quad \text{(III.2)}$$

$$\forall\, v \in W^{1,p(\cdot)}(\Omega)$$

If $u \neq 0$ we say that $\Lambda_{p(\cdot)}$ is an eigenvalue of (III.1), and that u is an eigenfunction corresponding to $\Lambda_{p(\cdot)}$.

Let $X := W^{1,p(\cdot)}(\Omega)$, and define the functionals $\mathscr{F}, \mathscr{G} : X \to \mathbb{R}$ by

$$\mathscr{F}(u) = \int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx \quad \text{(III.3)}$$

$$\textit{and} \quad \mathscr{G}(u) = \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx.$$

It is easy to see that $\mathscr{F}, \mathscr{G} \in C^1(X;\mathbb{R})$, and that for all $v \in X$ we have $\langle \mathscr{G}'(u), v \rangle_{X',X} = \int_\Omega |u|^{p(x)-2} uv\, dx$ and

$$\left\langle \mathscr{F}'(u), v \right\rangle_{X',X} = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \cdot \nabla v + |u|^{p(x)-2} uv)\, dx,$$

where $\langle \cdot, \cdot \rangle_{X',X}$ stands for the usual duality pairing of X and X' (the topological dual of X). Consider the level set $S_{\mathscr{G}} := \{u \in X \ : \ \mathscr{G}(u) = 1\}$, and the eigenvalue problem

$$\mathscr{F}'(u) = \mu \mathscr{G}'(u), \ \ u \in S_{\mathscr{G}}, \ \ \mu \in \mathbb{R}. \quad \text{(III.4)}$$

The existence of a sequence of nonnegative eigenvalues $\mu_n \to 0^+$ as $n \to \infty$ for the problem (III.4) was established in (17). It follows from the Ljusternik-Schnirelman theory (see, e.g., (4), (29)). We have $\mu_n = sup_{A \in \mathbb{A}_n} \inf_{u \in A} \mathscr{F}(u)$, with

$$\mathbb{A}_n := \{A \subset S_{\mathscr{G}} \ : \ \mathscr{F}(u) > 0 \text{ on } A,$$

$$A \text{ compact}, A = -A, \ \gamma(A) \geq n\},$$

where

$$\gamma(A) := inf\{k \in \mathbb{N} \ | \ \exists \ h : A \to \mathbb{R}^k \backslash \{0\},$$

$$h \textit{ odd and continuous}\}$$

is the genus of A. The eigenfunctions $u \in S_{\mathscr{G}}$ satisfy $\mathscr{F}'(u) = \mu \mathscr{G}'(u)$ or, equivalently, $\langle \mathscr{F}'(u), v \rangle_{X',X} = \mu \langle \mathscr{G}'(u), v \rangle_{X',X}$ for all $v \in X$. Hence,

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v\, dx = (\mu - 1) \int_\Omega |u|^{p(x)-2} uv\, dx$$

for all $v \in W^{1,p(\cdot)}(\Omega)$, which means that u is a weak solution of problem (III.1) with $\Lambda = \mu - 1$.

The following definition of viscosity solutions for second-order elliptic equations with fully nonlinear boundary conditions can be found in (3) (see also (5)).

Definition 2. Consider the boundary value problem

$$\begin{cases} F(x,u,Du,D^2 u) = 0 & \text{in } \Omega \\ H(x,u,Du) = 0 & \text{on } \partial\Omega. \end{cases} \quad \text{(III.5)}$$

(1) An upper semi-continuous function u is a viscosity subsolution of (III.5) if for every $\psi \in C^2(\overline{\Omega})$ such that $u - \psi$ has a maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ we have:

$$F(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0 \text{ if } x_0 \in \Omega,$$

and

$$\min\{H(x_0, \psi(x_0), D\psi(x_0)), F(x_0, \psi(x_0), D\psi(x_0),$$

$$D^2\psi(x_0))\} \leq 0 \text{ if } x_0 \in \partial\Omega.$$

(2) A lower semi-continuous function u is a viscosity supersolution of (III.5) if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have:

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0 \text{ if } x_0 \in \Omega,$$

and

$$\max\{H(x_0, \phi(x_0), D\phi(x_0)), F(x_0, \phi(x_0), D\phi(x_0),$$

$$D^2\phi(x_0))\} \geq 0 \text{ if } x_0 \in \partial\Omega.$$

(3) We say that a continuous function u is a viscosity solution of (III.5) if it is both a subsolution and a supersolution.

Remark 1. As remarked in (3), if $H(x,r,\cdot)$ is strictly increasing in the normal direction to $\partial\Omega$ at x, that is, for all $R > 0$ there exists $\nu_R > 0$ such that

$$H(x,r,\theta + \lambda\eta(x)) - H(x,r,\theta) \geq \nu_R \lambda \ \forall\ (x,r,\theta) \in \quad \text{(III.6)}$$

$$\in \partial\Omega \times [-R,R] \times \mathbb{R}^N \text{ and } \lambda > 0,$$

the definitions of viscosity sub and supersolutions for problem (III.5) in Definition **??** take a simpler form. Precisely,

(1) If u is a viscosity subsolution and $\psi \in C^2(\overline{\Omega})$ is such that $u - \psi$ has a maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ we have:

$$F(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0 \text{ if } x_0 \in \Omega,$$

and

$$H(x_0, \psi(x_0), D\psi(x_0)) \leq 0 \text{ if } x_0 \in \partial\Omega.$$

(2) If u is a viscosity supersolution and $\phi \in C^2(\overline{\Omega})$ is such that $u-\phi$ has a minimum at the point x_0 with $u(x_0) = \phi(x_0)$, then

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0 \text{ if } x_0 \in \Omega,$$

and

$$H(x_0, \phi(x_0), D\phi(x_0)) \geq 0 \text{ if } x_0 \in \partial\Omega.$$

Our next goal in this section is to prove that continuous weak solutions of (III.1) are, in fact, viscosity solutions (see Proposition 1. below). Before we proceed, we note that the Neumann eigenvalue problem (III.1) takes the form (III.5), with $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^{N\times N}_{\text{sym}} \to \mathbb{R}$ and $H : \partial\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$F(x,r,\theta,S) = -|\theta|^{p(x)-2}(Tr(S) + \ln|\theta| \langle \theta, \nabla p(x)\rangle) - \\ -(p(x)-2)|\theta|^{p(x)-4}\langle S\theta, \theta\rangle - \Lambda|r|^{p(x)-2}r$$

and

$$H(x,r,\theta) = \langle \theta, \eta \rangle,$$

where $\mathbb{M}^{N\times N}_{\text{sym}}$ is the space of $N\times N$ symmetric matrices, $\text{Tr}(S)$ stands for the trace of the matrix $S \in \mathbb{M}^{N\times N}_{\text{sym}}$, where $\langle \cdot,\cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. Note that the function H defined above satisfies the strict monotonicity condition in Remark 1 with $\nu_R = 1$, since in this case we have

$$H(x,r,\theta + \lambda\eta(x)) - H(x,r,\theta) = \langle \theta + \lambda\eta(x), \eta(x)\rangle - \\ -\langle \theta, \eta(x)\rangle = \lambda|\eta(x)|^2 \geq \lambda$$

for all $(x,r,\theta) \in \partial\Omega \times [-R,R] \times \mathbb{R}^N$ and $\lambda > 0$.

Proposition 1. Any continuous weak solution of (III.1) is also a viscosity solution of (III.1).

Let $u \in C(\overline{\Omega})$ be a weak solution of (III.1). To show that u is a viscosity supersolution of (III.1), let $x_0 \in \overline{\Omega}$, and consider a test function $\varphi \in C^2(\overline{\Omega})$ such that $u(x_0) = \varphi(x_0)$ and $u - \varphi$ has a minimum at x_0. If $x_0 \in \Omega$, we claim that we have

$$-\Delta_{p(x_0)}\phi(x_0) - \Lambda|\phi(x_0)|^{p(x_0)-2}\phi(x_0) \geq 0.$$

Indeed, if we assume that this inequality does not hold, then there exists $r > 0$ such that $B(x_0,r) \subset \Omega$ and

$$-\Delta_{p(x)}\phi(x) - \Lambda|\phi(x)|^{p(x)-2}\phi(x) < 0 \text{ for all } x \in B(x_0,r).$$

Taking r smaller, if necessary, we may assume that $u > \phi$ in $B(x_0,r)\setminus\{x_0\}$. Let

$$m = \inf_{x\in\partial B(x_0,r)} (u-\phi)(x) > 0,$$

and $\Phi(x) := \phi(x) + \frac{m}{2}$. Note that $\Phi(x_0) > u(x_0)$, $\Phi(x) < u(x)$ for all $x \in \partial B(x_0,r)$, and

$$-\Delta_{p(x)}\Phi(x) - \Lambda|\phi(x)|^{p(x)-2}\phi(x) < 0 \quad \text{for all } x \in B(x_0,r). \tag{III.7}$$

Multiply (III.7) by $(\Phi-u)^+$ and integrate over $B(x_0,r)$ to get

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)dx < \\ \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \Lambda|\phi|^{p(x)-2}\phi(\Phi-u)dx, \tag{III.8}$$

where we have used the fact that $(\Phi-u)^+ = 0$ on $\partial B(x_0,r)$. Extending $(\Phi-u)^+$ by zero outside $B(x_0,r)$, and using this extension as a test function in the weak formulation (III.2) gives

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi-u)dx = \\ \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \Lambda|u|^{p(x)-2}u(\Phi-u)dx. \tag{III.9}$$

After subtracting (III.9) from (III.8), using the fact that $u > \phi$ on $B(x_0,r)\setminus\{x_0\}$, and using the elementary inequality (see, e.g., Chapter I in (9))

$$|a-b|^p \leq 2^{p-1}\left(|a|^{p-2}a - |b|^{p-2}b\right)\cdot(a-b) \quad \text{for all } a,b \in \mathbb{R}^N \text{ and } p \geq 2, \tag{III.10}$$

we obtain

$$0 > \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx \\ + \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \Lambda\left(|u|^{p(x)-2}u - |\phi|^{p(x)-2}\phi\right)\cdot(\Phi-u)\,dx \\ \geq \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx \\ \geq \frac{1}{2^{p^+-1}} \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} |\nabla\Phi - \nabla u|^{p(x)}\,dx \geq 0,$$

which is clearly a contradiction. On the other hand, if $x_0 \in \partial\Omega$ we need to prove that

$$\max\left\{\frac{\partial \phi}{\partial \eta}(x_0), -\Delta_{p(x_0)}\phi(x_0) - \Lambda|\phi(x_0)|^{p(x_0)-2}\phi(x_0)\right\} \geq 0 \tag{III.11}$$

We proceed by contradiction. Assume that (III.11) does not hold. Then there exists $r > 0$ sufficiently small such that

$$\frac{\partial \phi}{\partial \eta}(x) < 0 \tag{III.12}$$

and

$$-\Delta_{p(x)}\phi(x) - \Lambda|\phi(x)|^{p(x)-2}\phi(x) < 0, \tag{III.13}$$

for all $x \in B(x_0, r)$. For $r > 0$ sufficiently small we have $u > \phi$ in $\left(\overline{B(x_0,r)} \setminus \{x_0\}\right) \cap \Omega$ and thus

$$m := \inf_{\partial B(x_0,r)\cap\overline{\Omega}} (u - \phi)(x) > 0.$$

With $\Phi(x) := \phi(x) + \frac{m}{2}$, note that $\Phi(x_0) > u(x_0)$, and that $\Phi(x) < u(x)$ for all $x \in \partial B(x_0, r) \cap \overline{\Omega}$. Multiplying (III.13) by $(\Phi - u)^+$ and integrating over $B(x_0, r) \cap \Omega$ gives

$$\begin{aligned} &\int_{B(x_0,r)\cap\Omega} |\nabla\phi|^{p(x)-2}\nabla\phi \cdot \nabla(\Phi - u)^+ \, dx - \\ &- \int_{\partial(B(x_0,r)\cap\Omega)} |\nabla\phi|^{p(x)-2}\frac{\partial\phi}{\partial\eta}(\Phi - u)^+ \, dx < \\ &< \int_{B(x_0,r)\cap\Omega} \Lambda|\phi|^{p(x)-2}\phi \cdot (\Phi - u)^+ \, dx. \end{aligned} \tag{III.14}$$

Since $(\Phi - u)^+ = 0$ on $\partial B(x_0, r) \cap \overline{\Omega}$, we have

$$\int_{\partial(B(x_0,r)\cap\Omega)} |\nabla\phi|^{p(x)-2}\frac{\partial\phi}{\partial\eta} \cdot (\Phi - u)^+ \, dx = \int_{B(x_0,r)\cap\partial\Omega} |\nabla\phi|^{p(x)-2}\frac{\partial\phi}{\partial\eta}(\Phi - u)^+ \, dx.$$

Thus,

$$\begin{aligned} &\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} |\nabla\Phi|^{p(x)-2}\nabla\Phi \cdot \nabla(\Phi - u) \, dx < \\ &< \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} |\nabla\phi|^{p(x)-2}\frac{\partial\phi}{\partial\eta}(\Phi - u) \, dx + \\ &+ \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|\phi|^{p(x)-2}\phi(\Phi - u) \, dx. \end{aligned} \tag{III.15}$$

Using the extension of $(\Phi - u)^+$ by zero outside $B(x_0, r) \cap \Omega$ as a test function in (III,2), we obtain

$$\begin{aligned} &\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\Phi - u) \, dx = \\ &= \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(\Phi - u) \, dx. \end{aligned} \tag{III.16}$$

Thus, subtracting (III.16) from (III.15) leads to

$$\begin{aligned} &\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right) \cdot \nabla(\Phi - u) \, dx \\ &< \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|\phi|^{p(x)-2}\phi - |u|^{p(x)-2}u\right) \cdot (\Phi - u) \, dx \\ &\quad + \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} |\nabla\phi|^{p(x)-2}\frac{\partial\phi}{\partial\eta}(\Phi - u) \, dx. \end{aligned}$$

Since $0 < r << 1$ was chosen such that (III.12) holds, we obtain that

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} |\nabla\phi|^{p(x)-2}\frac{\partial\phi}{\partial\eta}(\Phi - u) \, dx \leq 0.$$

Thus,

$$\begin{aligned} &\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right) \cdot \nabla(\Phi - u) \, dx \\ &< \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|\phi|^{p(x)-2}\phi - |u|^{p(x)-2}u\right) \cdot (\Phi - u) \, dx \leq 0, \end{aligned} \tag{III.17}$$

where the last inequality follows from the fact that $u \geq \phi$ on $B(x_0, r) \cap \Omega$. Applying (III.10) again, we deduce that

$$\begin{aligned} &\frac{1}{2^{p^- - 1}} \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} |\nabla\Phi - \nabla u|^{p(x)} \, dx \leq \\ &\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right) \cdot \nabla(\Phi - u) \, dx. \end{aligned} \tag{III.18}$$

Combining (III.17) and (III.18) gives

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} |\nabla\Phi - \nabla u|^{p(x)} \, dx < 0,$$

which is a contradiction. We conclude that u is a viscosity supersolution of (III.1). The proof of the fact that u is also a viscosity subsolution follows similarly.

IV THE ASYMPTOTIC BEHAVIOR OF THE SECOND EIGENVALUE/EIGENFUNCTION PAIRS

Consider a sequence of functions $\{p_n\} \subset C^1(\overline{\Omega})$ with

$$1 < p_n^- := min_{x\in\bar{\Omega}} p_n(x) \leq p_n^+ := \\ := max_{x\in\bar{\Omega}} p_n(x) < \infty, \forall\, n \in \mathbb{N}, \tag{IV.1}$$

and satisfying the following assumptions

$$p_n \to \infty \text{ uniformly in } \overline{\Omega}, \tag{IV.2}$$

$$\nabla \ln p_n \to \xi \text{ uniformly in } \overline{\Omega}, \tag{IV.3}$$

and

$$\frac{p_n}{n} \to q \text{ uniformly in } \overline{\Omega}, \tag{IV.4}$$

where $\xi \in C(\bar{\Omega},$, and $\mathbb{R}^N) q \in C(\bar{\Omega}, (0,+\infty))$ is such that $q^- := min_{x\in\bar{\Omega}} q(x) > 0$. Note that by (IV.4) we have

$$lim_{n\to\infty} \frac{p_n^-}{n} = q^-, lim_{n\to\infty} \frac{p_n^+}{n} = q^+ := max_{x\in\overline{\Omega}} q(x). \tag{IV.5}$$

It was shown in (17, Theorem 3.2) that the first eigenvalue of the $p(x)$-Laplacian with Neumann boundary condition is zero, and that the second eigenvalue is strictly greater than the first eigenvalue. It is also known that the eigenfunctions do not change sign in Ω. In this section we analyze the asymptotic behavior of the positive second eigenfunctions of the $p_n(x)$-Laplacian with Neumann boundary conditions:

$$\begin{cases} -\Delta_{p_n(x)} u = \Lambda_{p_n(\cdot)} |u|^{p_n(x)-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{IV.6}$$

as $n \to \infty$. In what follows, we will denote the positive second eigenvalues by Λ_n^2, and they are given by

$$\Lambda_n^2 = \frac{\int_\Omega |\nabla u_n|^{p_n(x)} dx}{\int_\Omega |u_n|^{p_n(x)} dx},\ n \in \mathbb{N}, \tag{IV.7}$$

where $u_n \in W^{1,p_n(\cdot)}(\Omega)$ is the eigenfunction associated to Λ_n^2, a minimizer of the functional

$$W^{1,p_n(\cdot)}(\Omega) \ni u \mapsto \int_\Omega \frac{1}{p_n(x)} |\nabla u|^{p_n(x)} dx$$

among all $u \in W^{1,p_n(\cdot)}(\Omega)$ satisfying the constraint $\int_\Omega \frac{1}{p_n(x)} |u|^{p_n(x)} dx = 1$. For each $n \in \mathbb{N}$, we define

$$c_n^2 := \inf\{\int_\Omega \frac{1}{p_n(x)} |\nabla u|^{p_n(x)} : u \in W^{1,p_n(x)}, \\ \int_\Omega \frac{1}{p_n(x)} |u|^{p_n(x)} dx = 1\}. \tag{IV.8}$$

Proposition 2. The sequence $\left\{ \left(\Lambda_n^2\right)^{\frac{1}{n}} \right\}$ is bounded.

Proof. Since

$$c_n^2 \leq \inf\left\{ \int_\Omega \frac{1}{p_n(x)} |\nabla u|^{p_n(x)} : u \in W_0^{1,p_n(x)},\ \int_\Omega \frac{1}{p_n(x)} |u|^{p_n(x)} dx = 1 \right\},$$

it follows from (25) that $\left\{ \left(c_n^2\right)^{\frac{1}{n}} \right\}$ is bounded.

Next, note that we have

$$\int_\Omega |u_n|^{p_n(x)} dx \geq \int_\Omega \frac{p_n^-}{p_n(x)} |u_n|^{p_n(x)} dx = p_n^- \int_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)} dx = p_n^-,$$

and thus, taking (IV.7) into account, we obtain

$$0 \leq (\Lambda_n^2)^{\frac{1}{n}} \leq \left(\frac{1}{p_n^-}\right)^{\frac{1}{n}} \left(\int_\Omega |\nabla u_n|^{p_n(x)} dx\right)^{\frac{1}{n}} \leq \\ \leq \left(\frac{1}{p_n^-}\right)^{\frac{1}{n}} \left(\int_\Omega \frac{p_n^+}{p_n(x)} |\nabla u_n|^{p_n(x)} dx\right)^{\frac{1}{n}} = \\ = \left(\frac{p_n^+}{p_n^-}\right)^{\frac{1}{n}} \left(\int_\Omega \frac{|\nabla u_n|^{p_n(x)}}{p_n(x)} dx\right)^{\frac{1}{n}} = \left(\frac{p_n^+}{p_n^-}\right)^{\frac{1}{n}} \left(c_n^2\right)^{\frac{1}{n}}$$

for all $n \in \mathbb{N}$. Since (IV.3) implies the existence of a positive constant $C > 0$ such that the Harnack type inequality $p_n^+ \leq C p_n^-$, $\forall\, n \in \mathbb{N}$ holds (see (22) for details), we have

$$lim_{n\to\infty} \left(\frac{p_n^+}{p_n^-}\right)^{\frac{1}{n}} = 1.$$

From the fact that the sequence $\left\{ \left(c_n^2\right)^{\frac{1}{n}} \right\}$ is bounded it now follows that $\left\{ \left(\Lambda_n^2\right)^{\frac{1}{n}} \right\}$ is also bounded, which concludes our proof.

Theorem 1.

Let $\{p_n\}$ be a sequence of variable exponents satisfying (IV.2)-(IV.4) and, for $n \in \mathbb{N}$, let Λ_n^2 and $u_n \in W^{1,p_n(\cdot)}(\Omega)$ be the be the second eigenvalue and the positive second eigenfunction corresponding to the Neumann problem (IV.6). Then there exists $\Lambda_\infty \in \mathbb{R}$ and $u_\infty \in C(\bar{\Omega}) \setminus \{0\}$ such that, after eventually extracting a subsequence, we have

$$\left(\Lambda_n^2\right)^{\frac{1}{n}} \to \Lambda_\infty \tag{IV.9}$$

and

$$u_n \to u_\infty \text{ uniformly in } \overline{\Omega}, \tag{IV.10}$$

as $n \to \infty$, where u_∞ is a nontrivial viscosity solution of the problem

$$\begin{cases} \min\{-\Delta_\infty u_\infty - |\nabla u_\infty|^2 ln|\nabla u_\infty|\langle \xi,\nabla u_\infty\rangle, \\ |\nabla u_\infty|^q - \Lambda_\infty |u_\infty|^q\} = 0 \quad \text{in} \quad \Omega \\ \frac{\partial u_\infty}{\partial \eta} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{IV.11}$$

Remark 2. At points where the gradient is vanishing, the PDE in (IV.11) is interpreted by assuming that the value of $v \mapsto |v|^2 \ln|v|$ at $v=0$ is zero.

Proof. Fix $m\in\mathbb{N}$ and choose $\varepsilon>0$ such that $\varepsilon<q^-$. We have $\frac{p_n^-}{n} > q^- - \varepsilon > 0$ and $n>m$ for all $n\in\mathbb{N}$ sufficiently large. In view of Hölder's inequality,

$$\begin{aligned}\int_\Omega |u_n|^{\frac{mp_n(x)}{n}}dx &\le \left(\int_\Omega |u_n|^{p_n(x)}dx\right)^{\frac{m}{n}}(|\Omega|)^{\frac{n-m}{n}} \\ &\le \left(\int_\Omega |u_n|^{p_n(x)}dx\right)^{\frac{m}{n}}(|\Omega+1|) \\ &\le \left(p_n^+\int_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)}dx\right)^{\frac{m}{n}}(|\Omega|+1) \\ &= (p_n^+)^{\frac{m}{n}}(|\Omega|+1).\end{aligned}$$

Since $lim_{n\to\infty}(p_n^+)^{\frac{m}{n}}=1$, we obtain that

$$\int_\Omega |u_n|^{\frac{mp_n(x)}{n}}dx \le 2(|\Omega|+1)$$

for $n\in\mathbb{N}$ sufficiently large. Using similar arguments we obtain, by Proposition 2, that there exists a constant $C=C(m)>0$ such that

$$\begin{aligned}\int_\Omega |\nabla u_n|^{\frac{mp_n(x)}{n}}dx &\le \left(\int_\Omega |\nabla u_n|^{p_n(x)}dx\right)^{\frac{m}{n}}(|\Omega|)^{\frac{n-m}{n}} \\ &\le \left(\Lambda_n^2\right)^{\frac{m}{n}}(p_n^+)^{\frac{m}{n}}(|\Omega|+1) \le C(m)\end{aligned}$$

for all $n\in\mathbb{N}$ sufficiently large. Combining these inequalities, and taking into account the fact that $n\in\mathbb{N}$ was chosen sufficiently large so that $\frac{mp_n(x)}{n} \ge \frac{p_n^-}{n} > q^- - \varepsilon$ in Ω, we deduce that the embedding $W^{1,\frac{mp_n(\cdot)}{n}}(\Omega)\subset W^{1,m(q^--\varepsilon)}(\Omega)$, and so the sequence $\{u_n\}$ is bounded in $W^{1,m(q^--\varepsilon)}(\Omega)$. If we now choose $m\in\mathbb{N}$ sufficiently large such that $m(q^--\varepsilon)>N$, it follows that the embedding of $W^{1,m(q^--\varepsilon)}(\Omega)$ into $C(\bar\Omega)$ is compact. Taking into account the reflexivity of the space $W^{1,m(q^--\varepsilon)}(\Omega)$, it follows that there exists a subsequence (not relabelled) of $\{u_n\}$ and a function $u_\infty\in C(\bar\Omega)$ such that $u_n \rightharpoonup u_\infty$ weakly in $W^{1,m(q^--\varepsilon)}(\Omega)$ and $u_n \rightharpoonup u_\infty$ uniformly in Ω.

Next, we prove that u_∞ is non-trivial. To this aim, recall that the second eigenfunctions satisfy the constraint $\int_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)}dx=1$, which gives

$$\left(\int_\Omega |u_n|^{p_n(x)}dx\right)^{\frac{1}{n}} \ge (p_n^-)^{\frac{1}{n}}. \tag{IV.12}$$

If $n\in\mathbb{N}$ is such that $\|u_n\|_\infty\le 1$, then $\|u_n\|_\infty^{p_n(\cdot)} \le \|u_n\|_\infty^{p_n^-}$ in Ω, and note that if $\|u_n\|_\infty>1$ we have $\|u_n\|_\infty^{p_n(\cdot)}\le\|u_n\|_\infty^{p_n^+}$ in Ω. Thus,

$$\int_\Omega |u_n|^{p_n(x)}dx \le \int_\Omega \|u_n\|_\infty^{p_n(x)}dx \le |\Omega|\max\left\{\|u_n\|_\infty^{p_n^-}, \|u_n\|_\infty^{p_n^+}\right\}.$$

Using (IV.12), we obtain

$$\max\left\{\|u_n\|_\infty^{p_n^-}, \|u_n\|_\infty^{p_n^+}\right\}^{\frac{1}{n}} \ge \left(\frac{p_n^-}{|\Omega|}\right)^{\frac{1}{n}}.$$

Letting $n\to\infty$ in the last inequality implies that $\max\left\{\|u_\infty\|_\infty^{q^-}, \|u_\infty\|_\infty^{q^+}\right\}\ge 1$, which shows that $u_\infty\ne 0$ in Ω.

In view of what we just shown, and taking again into account Proposition IV, we may extract a subsequence (not relabelled) such that (IV.9) and (IV.10) hold. The rest of the proof is devoted to showing that u_∞ is a viscosity solution of (IV.11).

Let $x_0\in\Omega$, and $\phi\in C^2(\Omega)$ be such that $u_\infty(x_0)=\phi(x_0)$ and $u_\infty-\phi$ has a minimum at x_0. The uniform convergence of u_n to u_∞ implies that there exists a sequence $\{x_n\}\subset\Omega$ such that $x_n\to x_0$, $u_n(x_n)=\phi(x_n)$, and $u_n-\phi$ has a minimum at x_n. Since for $n\in\mathbb{N}$ sufficiently large Proposition 1 implies that u_n is a continuous viscosity solution of (IV.6) with $\Lambda_{p_n(\cdot)}=\Lambda_n^2$, we have

$$\begin{aligned}|\nabla\phi(x_n)|^{p_n(x_n)-2}\left(\Delta\phi(x_n)+\ln|\nabla\phi(x_n)|\langle\nabla p_n(x_n),\nabla\phi(x_n)\rangle\right) \\ -(p_n(x_n)-2)|\nabla\phi(x_n)|^{p_n(x_n)-4}\Delta_\infty\phi(x_n) \\ \ge \Lambda_n^2|\phi(x_n)|^{p_n(x_n)-2}\phi(x_n).\end{aligned} \tag{IV.13}$$

We will need to study two cases. First, if $u_\infty(x_0)>0$, we have

$$\Lambda_n^2|\phi(x_n)|^{p_n(x_n)-2}\phi(x_n)=\Lambda_n^2|u_n(x_n)|^{p_n(x_n)-2}u_n(x_n)>0,$$

and thus, by (IV.13), we deduce that $|\nabla\phi(x_n)|>0$ for $n\in\mathbb{N}$ sufficiently large. Dividing both sides of (IV.13) by $(p_n(x_n)-2)|\nabla\phi(x_n)|^{p_n(x_n)-4}$, we find

$$\frac{-|\nabla\phi(x_n)|^2\left(\Delta\phi(x_n)+\ln|\nabla\phi(x_n)|\langle\nabla p_n(x_n),\nabla\phi(x_n)\rangle\right)}{p_n(x_n)-2}$$

$$-\Delta_\infty\phi(x_n) \geq \left(\frac{\left(\Lambda_n^2\right)^{1/n}|\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right)^n \frac{\phi(x_n)}{p_n(x_n)-2}.$$

Passing to the limit (supremum) as $n \to \infty$ and taking into account (IV.3) leads to

$$-\Delta_\infty\phi(x_0) - |\nabla\phi(x_0)|^2 \ln|\nabla\phi(x_0)|\langle\xi(x_0),\nabla\phi(x_0)\rangle$$
$$\geq \lim sup_{n\to\infty}\left[\left(\frac{(\Lambda_n^1)^{1/n}|\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right)^n \frac{\phi(x_n)}{p_n(x_n)-2}\right]. \quad \text{(IV.14)}$$

In particular, we have

$$-\Delta_\infty\phi(x_0) - |\nabla\phi(x_0)|^2 \ln|\nabla\phi(x_0)| \langle\xi(x_0),\nabla\phi(x_0)\rangle \geq 0. \quad \text{(IV.15)}$$

We claim that the following inequality holds.

$$|\nabla\phi(x_0)|^{q(x_0)} - \Lambda_\infty|\phi(x_0)|^{q(x_0)} \geq 0. \quad \text{(IV.16)}$$

Indeed, otherwise $|\nabla\phi(x_0)|^{q(x_0)} < \Lambda_\infty|\phi(x_0)|^{q(x_0)}$, and taking into account that

(IV.4) and (IV.9) imply

$$\lim_{n\to\infty}\left(\frac{(\Lambda_n^2)^{1/n}|\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right) = \frac{\Lambda_\infty|\phi(x_0)|^{q(x_0)}}{|\nabla\phi(x_0)|^{q(x_0)}} > 1, \quad \text{(IV.17)}$$

we deduce that there exists $\varepsilon > 0$ such that

$$\frac{(\Lambda_n^1)^{1/n}|\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}} \geq 1+\varepsilon$$

for all $n \in \mathbb{N}$ sufficiently large. Hence,

$$\limsup_{n\to\infty}\left(\left(\frac{(\Lambda_n^2)^{1/n}|\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right)^n \frac{\phi(x_n)}{p_n(x_n)-2}\right) \geq \lim_{n\to\infty}\frac{(1+\varepsilon)^n}{n}\left(\frac{\phi(x_n)}{\frac{p_n(x_n)-2}{n}}\right) = \infty,$$

which is a contradiction with (IV,14). Thus, (IV.16) holds, as claimed. Using (IV.15) and (IV.16) we deduce that in the case where $u_\infty(x_0) > 0$, we have

$$\min\{-\Delta_\infty\phi(x_0) - |\nabla\phi(x_0)|^2 \ln|\nabla\phi(x_0)| \langle\xi(x_0),\nabla\phi(x_0)\rangle, |\nabla\phi(x_0)|^{q(x_0)} - -\Lambda_\infty|\phi(x_0)|^{q(x_0)}\} \geq 0.(418) \quad \text{(IV.18)}$$

If $u_\infty(x_0) = \phi(x_0) = 0$, we either have $\nabla\phi(x_0) \neq 0$ (in which case we can use very similar arguments to conclude that (IV.15) and (IV.16) hold), or else $\nabla\phi(x_0) = 0$. For the latter, taking into account that $\Delta_\infty\phi(x_0) = 0$ and Remark 2, we arrive at (IV.15) again. On the other hand, (IV.16) is clearly also true. We conclude that (IV.18) holds.

Finally, let $x_0 \in \partial\Omega$, and assume that $u_\infty - \phi$ has a minimum at a point $x_0 \in \partial\Omega$ and $u_\infty(x_0) = \phi(x_0)$. Since u_n converges to u_∞ uniformly, we deduce that there exists $x_n \in \overline{\Omega}$ such that $x_n \to x_0$ and $u_n - \phi$ has a minimum point at x_n. Since u_n is viscosity supersolution of (III.1) we obtain, in view of Remark 1 that $\frac{\partial\phi}{\partial\eta}(x_n) \geq 0$, and hence

$$\frac{\partial\phi}{\partial\eta}(x_0) = \lim_{n\to\infty}\frac{\partial\phi}{\partial\eta}(x_n) \geq 0.$$

Hence, if $x_0 \in \partial\Omega$, we have

$$\max\{\min\{-\Delta_\infty\phi(x_0) - |\nabla\phi(x_0)|^2 ln|\nabla\phi(x_0)|\langle\xi(x_0),\nabla\phi(x_0)\rangle, |\nabla\phi(x_0)|^{q(x_0)} - \Lambda_\infty|\phi(x_0)|^{q(x_0)}\}, \frac{\partial\phi}{\partial\eta}(x_0)\} \geq 0.$$

Overall, we have shown that u_∞ is a viscosity supersolution of (IV.11). The proof of the fact that u_∞ is also a viscosity subsolution follows analogously. Therefore, u_∞ is a viscosity solution of (IV.11), which concludes the proof.

REFERENCES

[1] F. Abdullayev, M. Bocea, The Robin eigenvalue problem for the $p(x)$-Laplacian as $p \to \infty$. *Nonlinear Anal.* 91 (2013), 32-45

[2] E. Acerbi, G. Mingione, Regularity results for a class of functionals with non-standard growth, *Arch. Ration. Mech. Anal.* 156 (2001), 121-140.

[3] G. Barles, Fully non-linear Neumann type boundary conditions for second-order elliptic and parabolic equations. *J. Differential Equations* 106, (1993), 90-106.

[4] F. Browder, Existence theorems for nonlinear partial differential equations. In: *Global Analysis*, 1-60 (Proceedings of the Symposium Pure Mathematics, Vol. 16, Berkeley, California, 1968), Amer. Math. Soc., Providence, RI, 1970.

[5] M. G. Crandall, H. Ishii, and P.L. Lions, User's guide to viscosity solutions of second-order partial differential equations. *Bull. Am. Math. Soc.* 27, (1992), 1-67.

[6] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math* 66, (2006), 1383-1406.

[7] S.-G. Deng, Q. Wang, S. Cheng, On the $p(x)$-Laplacian Robin eigenvalue problem. *Appl. Math. Comput.* 217, No. 12 (2011), 5643-5649.

[8] S.-G. Deng, Eigenvalues of the $p(x)$-Laplacian Steklov problem, *J. Math. Anal. Appl.* 339, (2008), 925-937.

[9] E. DiBenedetto, *Degenerate Parabolic Equations*. Springer-Verlag, New York, 1993.

[10] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*. Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011.

[11] D. E. Edmunds, J. Lang, and A. Nekvinda, On $L^{p(x)}$ norms. *Proc. Roy. Soc. London Ser. A* 455, (1999), 219-225.

[12] D. E. Edmunds and J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$. *Proc. R. Soc. Lond. Ser. A* 437, (1992), 229-236.

[13] D. E. Edmunds and J. Rákosník, Sobolev embeddings with variable exponent. *Studia Math.* 143, (2000), 267-293.

[14] X.-L. Fan, Y.Z. Zhao, Q.-H. Zhang, A strong maximum principle for $p(x)$-Laplace equations. *Chinese J. Contemp. Math.* 24, (2003), 277-282.

[15] X.-L. Fan, Q.-H. Zhang, D. Zhao, Eigenvalues of the $p(x)$-Laplacian Dirichlet problem. *J. Math. Anal. Appl.* 302, (2005), 306-317.

[16] X.-L. Fan, Remarks on eigenvalue problems involving the $p(x)$-Laplacian. *J. Math. Anal. Appl.* 352, No. 1 (2009), 85-98.

[17] X.-L. Fan, Eigenvalues of the $p(x)$-Laplacian Neumann problems. *Nonlinear Analysis* 67, (2007), 2982-2992.

[18] G. Franzina and P. Lindqvist, An eigenvalue problem with variable exponents. *Nonlinear Anal.* 85, (2013), 1-16.

[19] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$. *Czechoslovak Math. J.* 41, (1991), 592-618.

[20] P. Lindqvist and T. Lukkari, A curious equation involving the ∞-Laplacian. *Adv. Calc. Var.* 3, No. 4 (2010), 409-421.

[21] J. J. Manfredi, J. D. Rossi, and J. M. Urbano, $p(x)$-harmonic functions with unbounded exponent in a subdomain. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26, No. 6 (2009), 2581-2595.

[22] J. J. Manfredi, J. D. Rossi, and J. M. Urbano, Limits as $p(x) \to \infty$ of $p(x)$-harmonic functions. *Nonlinear Anal.* 72, (2010), 309-315.

[23] M. Mihăilescu, *Eigenvalue Problems for Some Elliptic Partial Differential Operators*. Ph.D. Thesis, Central European University, Budapest, 2010. Available online at: http://web.ceu.hu/math/People/Alumni_and_Friends/Alumni/Mihai _MIHAILESCU_Thesis.pdf

[24] J. Musielak, *Orlicz spaces and modular spaces.* Lecture notes in mathematics, vol. 1034, Springer-Verlag, Berlin, 1983.

[25] M. Pérez-Llanos and J.D. Rossi, The behaviour of the $p(x)$-Laplacian eigenvalue problem as $p(x) \to \infty$. *J. Math. Anal. Appl.* 363 (2010), 502-511.

[26] M. Pérez-Llanos and J. D. Rossi, Limits as $p(x) \to \infty$ of $p(x)$-harmonic functions with non-homogeneous Neumann boundary conditions. In: *Nonlinear elliptic partial differential equations*, 187-201, Contemp. Math., Vol. 540, Amer. Math. Soc., Providence, RI, 2011.

[27] M. Ruzicka, *Electrorheological fluids: modeling and mathematical theory*. Springer-Verlag, Berlin (2002).

[28] L. Yu, F. Li, The stability of eigenvalues for the $p(x)$-Laplacian equations involving Robin boundary conditions, *J. Dyn. Con. Sys.* 24, (2018), 223-236.

[29] E. Zeidler, *Variational Methods and Optimization*. Nonlinear Functional Analysis and its Applications, vol. 3, Springer-Verlag, Berlin, 1990.

[30] V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.* 29, (1987), 33-36.