Acta of Turin Polytechnic University in Tashkent, 2021, 11, 47-50
Published Online June 2021 in Acta TTPU (http://www.acta.polito.uz/)

APPROXIMATE SOLUTION OF THE INVERSE CAUCHY PROBLEM FOR THE HEAT EQUATION BY THE QUASI-INVERSE METHOD

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Abstract– In this paper, the inverse Cauchy problem for the heat equation was considered, namely, its approximate solution is constructed by the quasi-inversion method.

Key words– inverse Cauchy problem, heat equation, quasi-inversion method, approximate solution, incorrect problem.

I Introduction

We consider the inverse Cauchy problem for the heat equation, construct an approximate solution of this problem by the quasi-inverse method using solutions of the Cauchy problem for the pseudo-parabolic equation.

Consider heat equation

$$u_t(x,t) = -u_{xx}(x,t) \tag{1}$$

in the $D = \{(x,t) : 0 < x < \pi, \ \ t > 0\}$.

Problem. Find the continuous in $\bar{D}$ function $u(x,t)$ which satisfies the equation (1) in the region D and following initial

$$u(x,0) = \varphi(x), \ 0 \le x \le \pi, \tag{2}$$

boundary conditions

$$\begin{matrix} u(0,t) = 0, \\ u(\pi,t) = 0, \end{matrix} \ \ 0 \le t \le T. \tag{3}$$

Problem (1) - (3) is called the inverse Cauchy problem for the heat equation. It is incorrect problem in the sense of Hadamard, which means a "small" change in the data can lead to a "big" change in the solution. Note that the inverse problem under consideration can be solved by many methods, such as the method of regularization of A. N. Tikhonov [1], the method of M. M. Lavrentyev [2], the method of quasi-solutions of V. K. Ivanov [3, 4], the quasi-inverse method by J. Lyons and so on. Most of the methods used to solve inverse problems of mathematical physics are also applicable to solving the classical (direct) problem of heat conduction.

The theoretical foundations and numerical solution of the inverse Cauchy problem for the heat equation have been considered by many authors. In the paper by Mu H., Li J. and Wang X [5] a cost function was constructed for the inverse heat problem and for its transformation into an optimization problem by the Tikhonov regularization method. P. Duda in his work [6] used the method of semi-discrete control to determine the temperature distribution in the plate. Kh. K. Al-Mahdawi in [7] solves the inverse Cauchy problem for the heat equation by the Picard method, which uses a regularizing family of operators $\{R_N\}$ that map the space $L_2[0,1]$ into itself.

In this paper, an approximate solution of the problem is constructed by replacing the considered problem with a problem of mathematical physics that is "close" to it in type. Section 2 presents the results of previous authors' work on the conditional correctness of this problem. In the next section, an auxiliary problem is presented and theorems on uniqueness and conditional stability are given. In the fourth section, the convergence of the solution of the auxiliary problem to the solution of original problem is proved.

An estimate of the conditional stability and uniqueness of the solution of this problem (1)-(3) follows from the works of S.G. Crane [8] and M. Landis [9].

II PRELIMINARY MATERIAL

Definition 1. By the solution of the problem we mean a continuous function in $\bar{D}$, a function having continuous derivatives participating in the equation, satisfying the equation in D and conditions (2) - (3).

If a solution to problem (1) - (3) exists, then it can be represented as

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n e^{n^2 t} \cdot \sin nx., \tag{4}$$

where $\varphi_n = \frac{1}{\pi}\int\limits_0^{\pi} \varphi(x) \sin nx dx$.

Here are some auxiliary lemmas.

Lemma 1. For the solution of the equation (1) at $t \in [0,T]$ the following estimate

$$\int\limits_0^{\pi} u^2(x,t)\,dx \le \left(\int\limits_0^{\pi} u^2(x,0)\,dx\right)^{1-\frac{t}{T}} \cdot \left(\int\limits_0^{\pi} u^2(x,T)\,dx\right)^{\frac{t}{T}} \tag{5}$$

is valid.

For the proof of the lemma see [10].

Let

$$M = \{u(x,t) : \|u(x,T)\| \le c\} \tag{6}$$

Theorem 1. If a solution to problem (1) - (3) exists and $u(x,t) \in M$, then it is is unique.

For the proof of the theorem see [8].

Let $u(x,t)$ be the solution of the problem (1) - (3) with exact data, and let $u_\varepsilon(x,t)$ be the solution of the problem (1) - (3) with approximate data.

Theorem 2. Let the solution of the problem (1) - (3) exists and $u(x,t), u_\varepsilon(x,t) \in M$, $\|\varphi(x) - \varphi_\varepsilon(x)\| \le \varepsilon$. Then for $U(x,t) = u(x,t) - u_\varepsilon(x,t)$ at $t \in (0;T)$ the estimate

$$\|U(x,t)\|^2 \le \varepsilon^{2\left(1-\frac{t}{T}\right)} \cdot (2c)^{\frac{2t}{T}} \tag{7}$$

is valid.

For the proof of the theorem see [8].

III THE CAUCHY PROBLEM FOR THE PSEUDO-PARABOLIC EQUATION

Problem. Find the continuous in $\bar{D}$ function $v(x,t)$ which satisfies the equation

$$v_t = \alpha v_{txx} - v_{xx},\ (x,t) \in D,\ \alpha > 0 \tag{8}$$

and conditions

$$v(x,0) = \varphi(x), \quad x \in [0,\pi], \tag{9}$$

$$\begin{matrix} v(0,t) = 0, \\ v(\pi,t) = 0, \end{matrix} \quad 0 \le t \le T. \tag{10}$$

If a solution to problem (8) - (10) exists, then it can be represented as

$$v(x,t) = \sum_{n=1}^{\infty} \varphi_n e^{\frac{n^2 t}{\alpha n^2 + 1}} \sin nx. \tag{11}$$

Lemma 2. Let in the domain D function $v(x,t)$ satisfies equation (8) and conditions (9)-(10), then for the solution $v(x,t)$ for any $x,t \in D$ the estimate

$$\int\limits_0^{\pi} v^2(x,t)\,dx \le \left(\int\limits_0^{\pi} v^2(x,0)\,dx\right)^{\frac{T-t}{T}} \left(\int\limits_0^{\pi} v^2(x,T)\,dx\right)^{\frac{t}{T}} \tag{12}$$

is valid.

Proof. Let $\psi(t)$ defined as follows

$$\psi(t) = \varphi^2 e^{\frac{2n^2 t}{\alpha n^2 + 1}}.$$

Let us calculate the first and second orders derivatives of the function $\psi(t)$

$$\psi'(t) = e^{\frac{2n^2 t}{\alpha n^2 + 1}} \frac{2n^2}{\alpha n^2 + 1} \varphi^2.$$

$$\psi''(t) = e^{\frac{2n^2 t}{\alpha n^2 + 1}} \frac{4n^4}{(\alpha n^2 + 1)^2} \varphi^2.$$

Denote $\gamma(t) = \ln \psi(t)$. Then

$$\gamma''(t) = \frac{\psi \cdot \psi'' - {\psi'}^2}{\psi^2} \ge 0$$

That from the last inequality

$$\gamma(t) \le \gamma(0)\frac{T-t}{T} + \gamma(T)\frac{t}{T}.$$

When going back to $\psi(t)$ we get

$$\psi(t) \le (\psi(0))^{\frac{T-t}{T}} (\psi(T))^{\frac{t}{T}}$$

Hence

$$\int\limits_0^{\pi} v^2(x,t)\,dx \le \left(\int\limits_0^{\pi} v^2(x,0)\,dx\right)^{\frac{T-t}{T}} \left(\int\limits_0^{\pi} v^2(x,T)\,dx\right)^{\frac{t}{T}}.$$

Theorem 3. The solution of the problem (8)-(10) exists and it is unique.

Proof. Existence. Consider series (11). To determine the necessary and sufficient conditions for the existence of a solution to problem (8)-(10), it is necessary to prove the uniform convergence of the following series, which are derivatives of the functional series (11) participating in the equation (8):

$$v_t(x,t) = \sum_{n=1}^{\infty} \frac{n^2 \varphi_n e^{\frac{n^2 t}{\alpha n^2+1}} \sin nx}{\alpha n^2+1}, \quad (13)$$

$$v_{txx}(x,t) = \sum_{n=1}^{\infty} \frac{-n^4 \varphi_n e^{\frac{n^2 t}{\alpha n^2+1}} \sin nx}{\alpha n^2+1}, \quad (14)$$

$$v_{xx}(x,t) = \sum_{n=1}^{\infty} -n^2 \varphi_n e^{\frac{n^2 t}{\alpha n^2+1}} \sin nx. \quad (15)$$

For all three series, we define the majorant series as follows

$$\sum_{n=1}^{\infty} n^2 |\varphi_n|. \quad (16)$$

For the uniform convergence of series (12) - (14), it suffices to show the convergence of the general majorant series (15). The proof of the convergence of series (15), in turn, reduces to the proof of the convergence of the following series

$$\sum_{n=1}^{\infty} n^2 |\varphi_n|. \quad (17)$$

From the known properties of the Fourier series (11), we obtain the requirements for the function $\varphi(x)$:

The derivatives of the function $\varphi(x)$ are continuous up to the second order inclusive, the third derivative is piecewise continuous and

$$\varphi(0) = \varphi(\pi) = 0, \qquad \varphi''(0) = \varphi''(\pi) = 0. \quad (18)$$

Uniqueness. Let there exist two solutions of problem (8) - (10) which satisfy equation (8) with the same conditions (9) - (10). Then it is obvious that $v = v_1 - v_2$ also satisfies equation (8) with homogeneous initial and boundary conditions. The corresponding problem has the form

$$v_t = \alpha v_{txx} - v_{xx}, \alpha > 0, (x,t) \in D \quad (19)$$

$$v(x,0) = 0, \quad x \in [0,\pi] \quad (20)$$

$$\begin{matrix} v(0,t) = 0, \\ v(\pi,t) = 0, \end{matrix} \quad 0 \le t \le T. \quad (21)$$

According to (19) - (21) and the inequality from Lemma 1, we have $\int_0^{\pi} v^2(x,t)\,dx \le 0$ or $v(x,t) = 0$. Hence $v_1 = v_2$, i.e. the solution to the problem is unique.

IV APPROXIMATE SOLUTION BY THE QUASI-INVERSION METHOD

Obviously, the family of operators B_α defined by the formula $u_\alpha(x,t) = B_\alpha \varphi(x)$ is a regularizing family with respect to the Cauchy problem.

Let us estimate the efficiency of the regularizing family.

It is easy to see that

$$\|B_\alpha\| = \max_n e^{\frac{n^2 t}{\alpha n^2+1}} \le e^{\frac{t}{\alpha}} \quad (22)$$

As an approximate solution based on approximate data, consider the function

$$u_{\alpha_\varepsilon}(x,t) = \sum_{n=1}^{\infty} \varphi_{n_\varepsilon} e^{\frac{n^2 t}{\alpha n^2+1}} \sin nx, \quad (23)$$

where $\varphi_{n_\varepsilon} = \frac{2}{\pi} \int_0^{\pi} \varphi_\varepsilon(x) \sin nx dx, \quad n = 1, 2, ...$

Let us estimate in the norm the difference between the exact solution and the regularized solution from the approximate data

$$\|u(x,t) - u_{\alpha_\varepsilon}(x,t)\| \le \|u(x,t) - u_\alpha(x,t)\| + \|u_\alpha(x,t) - u_{\alpha_\varepsilon}(x,t)\|. \quad (24)$$

Let us estimate the first term on the right-hand side of inequality (14)

$$\|u_\alpha(x,t) - u(x,t)\|^2 = \sum_{n=1}^{\infty} \varphi_n^2 \left(e^{n^2 t} - e^{\frac{n^2 t}{\alpha n^2+1}} \right)^2 \quad (25)$$

at

$$\|u(x,T)\|^2 = \sum_{n=1}^{\infty} \varphi_n^2 e^{2n^2 T} \le c^2. \quad (26)$$

To find the conditional extremum, we will use the Lagrange multiplier method.

Let us construct the Lagrange function

$$F(\varphi_n, \lambda) = \sum_{n=1}^{\infty} \varphi_n^2 \left(e^{n^2 t} - e^{\frac{n^2 t}{\alpha n^2+1}} \right)^2 + \lambda \left(\sum_{n=1}^{\infty} \varphi_n^2 e^{2n^2 T} - c^2 \right).$$

From $\nabla F(\varphi_n, \lambda) = 0$ we have that the value of the estimated norm does not exceed the maximum of the following expression

$$\varphi_n \left(e^{n^2 t} - e^{\frac{n^2 t}{\alpha n^2 + 1}} \right)$$

$$\varphi_n e^{n^2 T} \leq c.$$

That is, does not exceed the maximum of the function

$$\gamma(n) = c e^{-n^2 (T-t)} \left(1 - e^{-\frac{n^4 t \alpha}{\alpha n^2 + 1}} \right)$$

.

Find the extremum of the function $\gamma(n)$.

$$\gamma(n) \leq c \frac{n^4 t \alpha}{\alpha n^2 + 1} e^{-n^2 (T-t)} \leq c \cdot n^4 t \alpha \cdot e^{-n^2 (T-t)} = \beta(n)$$

$$\beta'(n) = c t \alpha n^3 \left[4 - 2n^2 (T-t) \right] e^{-n^2 (T-t)}$$

Hence it follows that

$$n^2 = \frac{2}{T-t}.$$

Then

$$\max_n \gamma(n) \leq \max_n \beta(n) = \frac{4ct}{(T-t)^2} \alpha e^{-2}. \quad (27)$$

Let us estimate the second norm on the right side of the inequality (24)

$$\| u_\alpha(x,t) - u_{\alpha_\varepsilon}(x,t) \|^2 = \sum_{n=1}^{\infty} e^{\frac{2n^2 t}{\alpha n^2 + 1}} (\varphi_n - \varphi_{n_\varepsilon})^2 \leq e^{\frac{2t}{\alpha}} \varepsilon^2. \quad (28)$$

Considering estimates (27) and (28) we obtain

$$\| u(x,t) - u_{\alpha_\varepsilon}(x,t) \| \leq \| u(x,t) - u_\alpha(x,t) \| + \| u_\alpha(x,t) - u_{\alpha_\varepsilon}(x,t) \| \leq \frac{4ct}{(T-t)^2} \alpha e^{-2} + e^{\frac{2t}{\alpha}} \varepsilon^2. \quad (29)$$

From the right-hand side of inequality (29), for any $\varepsilon > 0$, we find

$$\inf \left\{ \frac{4ct}{(T-t)^2} \alpha e^{-2} + e^{\frac{2t}{\alpha}} \varepsilon^2 \right\} = \psi(\varepsilon), \; t \neq T$$

and find α.

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