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GENERALIZATION OF THE RIEMANN METHOD FOR SOLVING THE PROBLEM OF WAVE PROPAGATION IN A SEMI-INFINITE GAS PIPELINE BY A QUADRATIC LAW OF HYDRAULIC RESISTANCE

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Abstract– A mathematical model of the problem of pulse propagation in a semi-infinite gas pipeline was compiled by expressing the pressure drop by a quadratic law of resistance and by the local component of the gas inertia force in the law of conservation of momentum and using the law of conservation of mass in a one-dimensional formulation. The model repeats the Riemann problem but takes into account the frictional resistance force.

Key words– quadratic law of resistance, semi-infinite gas pipeline, pulse propagation, quasi-one-dimensional equations for pipeline transportation of real gas, finite-difference method, computational experiment.

I Introduction

The study in [1] is devoted to the modeling and numerical calculation of the gas distribution pipeline network with a special focus on the elementary sections of the gas pipeline. Gas flues are the most important component of such systems as they determine the main dynamic characteristics. Isothermal one-way flow is assumed when simulating gas flow through a gas flue.

Trunk gas pipelines are the main part of the gas transport system [2]. The main share of energy consumption in pipeline gas transportation falls on this part of the system. The characteristics of the pipeline network, along with the set technological objectives, are the determining factors for the operating mode of the system equipment, located mainly at the compressor stations. Simple analytical formulas were obtained to determine the mass flow rate, pressure and supercompressibility coefficient [3]. The distribution of the gas mass flow rate between the parallel lines with a variable gas supercompressibility coefficient occurs similar to the case of a constant gas supercompressibility coefficient. The difference lies in the definition of the inlet and outlet pressure values, i.e. when solving transcendental equations.

In [4], a new approximate model of gas flow in pipelines was developed, which makes it possible to calculate the processes of unsteady gas flow in gas pipelines considering the flow inertia. A comparison of the results of calculations for this model with the exact analytical solution of the original linearized system of equations for gas transport was conducted.

The problems with the joint consideration of the inertia force in the form of the local component and the resistance force expressed by the quadratic law are of particular interest. These problems are not described by the "long" and "short" pipeline approaches but in partial cases, they can lead to the results inherent in these approaches.

Below we will dwell on such problems when a boundary condition is given for $x=0$ and $t=0$, conditions are set in the form of directional changes in the gas velocity and its first time derivative. The domain under consideration is the left semi-axis $x,(x\geq 0)$.

With the auxiliary function and the transition to traveling waves, a general solution of the problem is obtained with the participation of the resistance parameter and some support function u^0 , which is the solution to the Riemann problem without considering the friction force. Further, with boundary conditions, a solution is obtained with respect to the gas velocity, which, as $\varepsilon\to 0$, passes to the well-known Riemann solution.

II SOLUTION OF THE PROBLEM. METHODS

Ignoring the convective component of the gas inertia force and the route slope from the horizon, the quasi-one-dimensional equations of conservation of momentum and mass have the form [5, 6]:

$$\begin{cases} -\frac{\partial p}{\partial x} = \rho \frac{\partial u}{\partial t} + \varepsilon \rho u^2 \\ \\ -\frac{\partial \rho}{\partial t} = \frac{\partial \rho u}{\partial x} \end{cases} \tag{1}$$

Here x,t are the distance and time; u,p,ρ are the average values of gas velocity, pressure and density in section x at time point t; $\varepsilon = \frac{\lambda}{2D}$ is the parameter of the resistance force by the Darcy-Weisbach law; λ, D are the friction resistance coefficient and pipeline diameter.

In the absence of external disturbances, we can pose the following boundary value problem with the following boundary condition

$$u(0,t) = \mu(t) \tag{2}$$

and initial conditions

$$u(x,0) = \phi(x), \frac{\partial u(x,0)}{\partial t} = \psi(x) \tag{3}$$

Thus, the problem of the gas-dynamic state of a semi-infinite pipeline ($x \geq 0$) is solved in this article.

Let us introduce the propagation velocity of small disturbances $c = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{ZRT}$ in the real gas flow (Z,R,T are the supercompressibility coefficient, reduced gas constant and average gas temperature) transported through the pipeline and the following auxiliary function

$$\varphi = \ln \frac{\rho}{\rho_*} \tag{4}$$

here ρ_* is the characteristic gas density. Ignoring the term $u\frac{\partial \varphi}{\partial x}$ in the second equation, which at low hydrodynamic velocities and insignificant disturbances of density is a small value, allows us to write system (1) in the following form:

$$\begin{cases} \frac{\partial u}{\partial t} + c^2 \frac{\partial \varphi}{\partial x} = -\varepsilon u^2 \\ \frac{\partial \varphi}{\partial t} + \frac{\partial u}{\partial x} = 0 \end{cases} \tag{5}$$

In numerous literary sources, including textbooks [7], this system is analyzed and solved without the right-hand side of the first equation.

In some cases, calibration functions are introduced [8]

$$A = u + c\varphi, B = u - c\varphi \tag{6}$$

Then system (7) takes the form:

$$\begin{cases} \frac{\partial A}{\partial t} + c\frac{\partial A}{\partial x} = -\varepsilon u^2 \\ \frac{\partial B}{\partial t} - c\frac{\partial B}{\partial x} = -\varepsilon u^2 \end{cases} \tag{7}$$

With the involvement of the substantial derivative, system (8) can be written as:

$$\begin{cases} \frac{dA}{dx} = -\frac{\varepsilon u^2}{c}, \\ \frac{dB}{dx} = -\frac{\varepsilon u^2}{c}. \end{cases}$$

From this system of equations, we can compose an equation relative to the velocity $u(x,t) = A(\eta,t) + B(\xi,t)$,

$$\begin{cases} \frac{du}{u^2} = -\frac{\varepsilon dx}{c}, \\ \frac{dx}{dt} = c \end{cases}$$

The first integral in x of the first equation of this system, written as $\frac{d}{dx}\left(\frac{1}{u}\right) = \frac{\varepsilon dx}{c}$, is

$$\frac{1}{u} = \frac{1}{u^0} + \frac{\varepsilon x}{c}$$

where $u^0 = u(\eta,\xi)|_{\varepsilon=0}$ is the solution to the homogeneous system (1), i.e., for $\varepsilon = 0$

Hence, we find the reference solution to the problem

$$u = \frac{u^0}{1 + \frac{\varepsilon x}{c}u^0}$$

This solution to the problem implies the following general pattern in the approximation of a long pipeline: the solution to the problem with allowance for friction is less than the solution to the problem without considering the friction.

It is necessary to determine the value of u^0 . Let us consider the well-known solution of the Cauchy problem from mathematical physics [9]:

$$u^0(x,t) = \frac{\Phi(x+ct) - \Phi(x-ct)}{2} + \frac{1}{2\pi}\int_{x-ct}^{x+ct} \Psi(a)da \tag{8}$$

where Φ and Ψ are the sought-for value and its time derivative at .

To have zero sought-for value, it is necessary to assume that

$$\Phi(x) = \begin{cases} \phi(x) for x > 0 \\ -\phi(x) for x < 0 \end{cases}$$

and

$$\Psi(x) = \begin{cases} \psi(x) for x > 0 \\ -\psi(x) for x < 0 \end{cases}$$

In this case, $u^0(0,t) = 0$ holds, and the condition for $x = 0$ can be used. In addition, due to the fulfillment of the following conditions

$$u^0(x,0) = \Phi(x) = \phi(x), \frac{\partial u^0(x,0}{\partial t} = \Psi(x) = \psi(s)$$

for $x > 0$, the solution satisfies two initial conditions.

Then, taking into account the velocity perturbation in section , we can write the solution to the problem in the following form

$$u^0(x,t) = \begin{cases} \mu(t - \frac{x}{c}) + \frac{\phi(x+ct)-\phi(x-ct)}{2} + \frac{1}{2a}\int_{x-ct}^{x+ct} \psi(a)da \quad for \\ x < tc \\ \frac{\phi(x+ct)-\phi(x-ct)}{2} + \frac{1}{2a}\int_{x-ct}^{x+ct} \psi(a)da \quad for \\ x > tc \end{cases}$$

Substitution of value u^0 into equation (9) leads to the following result for the gas velocity:

$$u(x,t) = \begin{cases} \frac{\mu(t-\frac{x}{c})+\frac{1}{2}[\phi(x+ct)-\phi(x-ct)]+\frac{1}{2}\int_{x-ct}^{x+ct}\psi(a)da}{1+\frac{\varepsilon x}{c}\{\mu(t-\frac{x}{c})+\frac{1}{2}[\phi(x+ct)-\phi(x-ct)]+\frac{1}{2}\int_{x-ct}^{x+ct}\psi(a)da\}} \quad for \\ x \leq tc \\ \frac{\frac{1}{2}[\phi(x+ct)-\phi(x-ct)]+\frac{1}{2}\int_{x-ct}^{x+ct}\psi(a)da}{1+\frac{\varepsilon x}{c}\{\frac{1}{2}[\phi(x+ct)-\phi(x-ct)]+\frac{1}{2}\int_{x-ct}^{x+ct}\psi(a)da\}} \quad for \\ x \leq tc \end{cases}$$

Thus, an analytical solution to the formulated problem is obtained with respect to the hydrodynamic gas velocity.

This solution generalizes the well-known solution of the Riemann problem taking into account the quadratic law of resistance in a gas pipeline. Indeed, if we accept $\varepsilon = 0$, then we obtain the classical Riemann solution [9] on a one-way wave.

III MATERIALS

The analytical solution obtained applies only to the gas flow rate. To present the complete picture, it is necessary to find a solution to the pressure problem, for which one can turn to the numerical method.

The transition to discrete coordinates is performed so that the pulse jump occurs at the calculated node. If the step in space is h , then it takes time h/c for the wave to travel this distance. Accordingly, the time step is $\tau = h/c$. For example, if $h = 1m$ and $c = 400m/c$, then $\tau = h/c = 0.0025s$. This choice of discrete coordinate steps makes it possible to determine the value of the pressure jump at the front of the shock wave. To confirm this statement, let us turn to the next problem.

Problem 1. We assume that before the onset of disturbances, the gas is at rest, and at $t = 0$, at the inlet to the section, the gas velocity instantly increases to U and then the constant value of the velocity at the inlet is maintained. Accordingly, the initial pressure value and the pressure value in the unperturbed region is P. In accordance with this, the initial value of the density and the value of gas density in the unperturbed region is $\rho_0 = P/c^2$. That is, ρ_0 can be taken for the characteristic density of gas. The auxiliary function is $\ln\frac{\rho}{\rho_0} = \ln\frac{\rho c^2}{P}$. In the unperturbed calculation zone, we have $u = 0, \varphi = 0$.

Let us assume that the impulse wave has reached node n in time, which corresponds to the coordinate $x = nh$. Th gas velocity at this point is

$$u_n^n = \frac{U}{1+\frac{\varepsilon nh}{c}U}$$

At the nodes along x following the front of the disturbance, the gas is at rest, in particular:

$$u_{n+1}^n = 0$$

Then from the second equation of system (6) it follows

$$\frac{\varphi_n^n - \varphi_n^{n-1}}{\tau} = -\frac{u_{n+1}^n - u_n^n}{h}$$

Here, we used the backward pattern in time, and the forward pattern along the x coordinate. Since the gas is at rest, the equations $u_{n+1}^n = 0, \varphi_n^{n-1} = 0$ are appropriate. In this regard, it follows from the last finite difference equation

$$\varphi_n^n = \frac{\tau}{h}u_n^n$$

considering $\tau = \frac{h}{c}$ we obtain

$$\ln\frac{\rho_n^n}{\rho_0} = \ln\frac{P_n^n}{P} = \frac{u_n^n}{c}$$

By potentiating these relations, we obtain the density values

$$\rho_n^n = \rho_0 exp\left(\frac{U}{c+\varepsilon nhU}\right)$$

and the pressure values

$$P_n^n = Pexp\left(\frac{U}{c+\varepsilon nhU}\right)$$

at the disturbance front for the n-th time step.

Since $x^* = nh$ is the coordinate of the wave front, the last relations can be written in the form of the following formulas

$$u(x^*) = \frac{cU}{c+\varepsilon x^* U}$$

$$\rho(x^*) = \rho_0 exp\left(\frac{U}{c+\varepsilon x^* U}\right)$$

$$p(x^*) = Pexp\left(\frac{U}{c+\varepsilon x^* U}\right)$$

From the last formula, it follows that the highest pressure value is

$$p_0^* = Pexp(U/c)$$

that is, a jump by $exp(U/c)$ times is expected at the beginning of the process, and then, as the disturbance propagates, the pressure jump decreases.

Taking into account the obtained value of φ_n^n at the disturbance front and the known values of u_i^n and u_i^{n-1} , let us turn to the first equation of system (6), represented in a discrete form:

$$\frac{\varphi_{i+1}^n - \varphi_i^n}{h} = -\frac{1}{c^2}\left[\frac{u_{i+1}^n - u_{i+1}^{n-1}}{\tau} + \varepsilon(u_{i+1}^n)^2\right]$$

Here it was taken into account that the pressure change occurs against the direction of propagation of the pulse wave.

The last finite difference equation implies the recurrence formula

$$\varphi_i^n = \varphi_{i+1}^n + \frac{h}{c^2}\left[\frac{u_{i+1}^n - u_{i+1}^{n-1}}{\tau} + \varepsilon(u_{i+1}^n)^2\right]$$

which allows calculating the values of the auxiliary function for $i = n-1...0$.

The length of the calculated section is bounded and equals to $l = Nh$. For $n = N$, when the wave reaches the end of the calculated section, the velocity is

$$u_N^N = \frac{U}{1+\frac{\varepsilon Nh}{c}U}$$

Accordingly, the limit value of the auxiliary function is

$$\varphi_N^N = \frac{\tau}{h}u_N^N$$

Using the above recurrent formula, we calculate the values of the auxiliary function for $i = N-1...0$.

For , i.e. when the wave leaves the computational domain, the value of the auxiliary function at the computation boundary can be found from the second equation of system (6), presented in a discrete form:

For $n > N$, i.e. when the wave leaves the computational domain, the value of the auxiliary function at the computation boundary can be found from the second equation of system (6), presented in a discrete form:

$$-\frac{u_{N+1}^n - u_N^n}{h} = \frac{\varphi_N^n - \varphi_N^{n-1}}{\tau}$$

Here $u_N^n = \frac{U}{1+\frac{\varepsilon(n+1)h}{c}U}$

Hence, we determine

$$\varphi_N^n = \varphi_N^{n-1} - \frac{\tau}{h}(u_{N+1}^n - u_N^n)$$

Further, realizing the recurrence formula, we calculate φ_i^n for $i = N-1...0$.

Having calculated the values of u_i^n by the above formula and the values of auxiliary function φ_i^n by the marching method, the fields of the velocities and auxiliary function are determined. The gas density in grid coordinates is determined by the following formula

$$\rho_i^n = \rho_0 exp(\varphi_i^n)$$

and the gas pressure is determined according to the formula

$$p_i^n = c^2\rho_i^n.$$

If necessary, the mass flow rate of gas can be calculated by the formula

$$M_i^n = f\rho_i^n u_i^n,$$

where $f = \pi D^2/4$ is the cross-sectional area of the pipeline.

IV Calculation results

A computational experiment was conducted for a pipeline with a diameter of 1 m, a resistance coefficient of 0.01 for ε=0.005 m^{-1}. We considered the case when the initial state is rest; at the beginning of the section, the velocity increases by a jump from 0 m/s to 20 m/s. The speed of sound is 378.21 m/s.

Fig. 1 shows the contours of the gas velocity in the coordinate plane.

The inlet velocity jumps from 0 m/s to 20 m/s. The diagonal represents the pulse jump front; the lower right part is the unperturbed zone. The velocity along the flow length decreases by a hyperbolic law. At the jump front, the velocity values are: for tc/l=0.1 - 15.8177 m/s, for tc/l=0.4 - 9.7199 m/s, tc/l=0.1 - 5.4884 m/s.

Fig. 2 shows the isolines of the gas mass flow rate in the coordinate plane tc/l=0.1 .

The diagonal represents the pulse jump front; the lower right part is an unperturbed zone. For ct/l=0.01 , the mass flow rate at the inlet to the section is 12.190 kg/s. For x=0.1 km it decreases to 11.264 kg/s. Then it decreases by a jump to 0 kg/s. Over time, the inlet mass flow rate decreases: for tc/l=0.50 - 11.844 kg/s, for tc/l=1.00 - 11.746 kg/s, and for tc/l=2.00 - 11.700 m/s.

With distance, the gas mass flow rate decreases practically by the hyperbolic law.

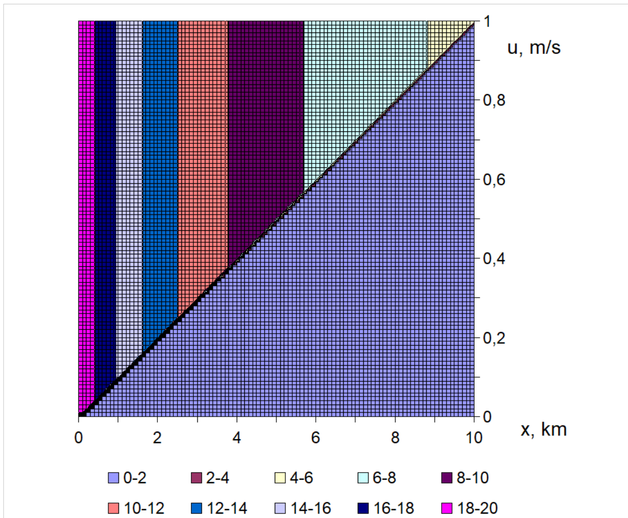

Fig. 1: Isolines of the average gas velocity in plane (x, ct/l).

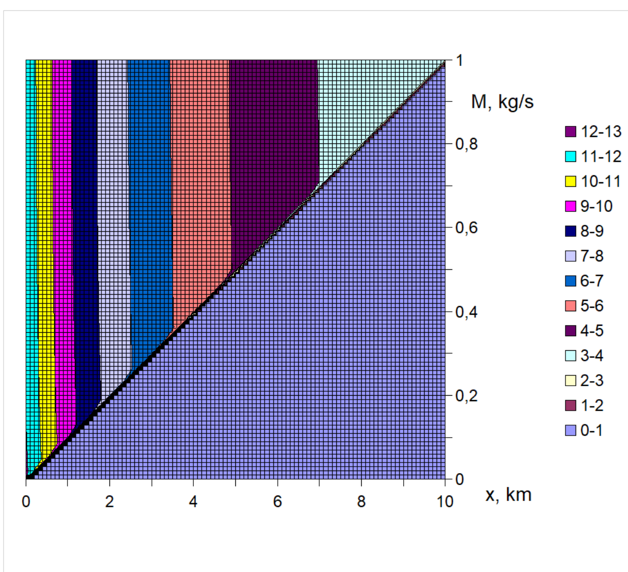

Fig. 2: Isolines of mass flow rate in plane (x, ct/l).

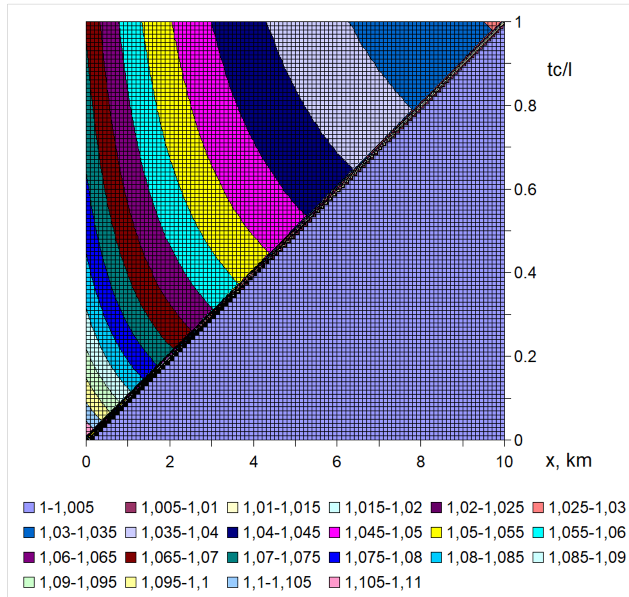

Fig. 3: Field of isobars in plane (x, ct/l).

Fig. 3 shows the field of isobars in plane $(x, ct/l)$.

The calculations were conducted in steps $h = 0.0001l$ and $\tau = \frac{h}{c}$. For visualization, we used every tenth of the length and dimensionless time values. For tc/l=0.01 , the inlet gas pressure was 0.11100 MPa, at the next step along the length, it was 0.10529 MPa. Then, the pressure decreased by a jump to 0.10000 MPa. For tc/l=0.1 , the inlet pressure was 0.10993 MPa, and at the jump front it was 0.10427 MPa. For tc/l=0.5 , the inlet pressure was 0.10785 MPa, and at the jump front it was 0.10230 MPa. When the wave reached the end of the section, the pressure at the front was 0.10146 MPa, and at the inlet, it was 0.10696 MPa

Problem 2. The gas pipeline is at rest, i.e., the initial gas velocity is zero. The initial pressure is $p(0,0) = p_{00}$. The initial density distribution is homogeneous and, using an auxiliary function, it is expressed as $\varphi(x,0) = 0$. At the time point $t = 0$, the gas velocity at the inlet to the gas pipeline changes by the sinusoidal law

$$u(0,t) = U \sin 0.1\pi t$$

Let us investigate the dynamics of the impulse propagation process.

We use the formula derived to calculate the velocity. To calculate the pressure field, the value of the pressure in the unperturbed zone is known.

To calculate the value of the auxiliary function at the disturbance front, which left the calculation domain, we used the approximation of the second equation of system (6).

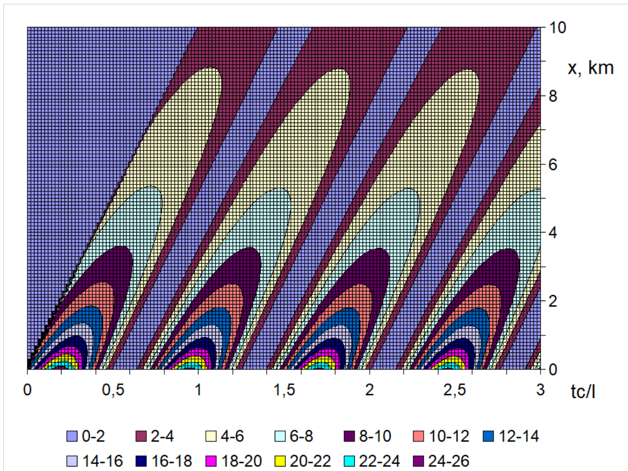

Fig. 4: Isolines of the average gas velocity in plane (x, ct/l) for $u(0,t) = 2\sin 0.1\pi tm/s$.

$$\frac{\varphi_n^n - \varphi_n^{n-1}}{\tau} = -\frac{u_{n+1}^n - u_n^n}{h}$$

Here for $i > n$, i.e. in the unperturbed zone, $u_{n+1}^n = 0$. Similarly, we can accept φ_n^{n-1}. So,

$$\varphi_n^n = \frac{\tau}{h} u_n^n$$

Further, according to the values obtained by the formula

$$\varphi_i^n = \varphi_{i+1}^n + \frac{h}{c^2}\left[\frac{u_{i+1}^n - u_{i+1}^{n-1}}{\tau} + \varepsilon (u_{i+1}^n)^2\right]$$

we calculate the values of φ_i^n for $i = n-1...0$. For $i \le N$ we save the values φ_i^n of for future use, i.e. for calculating and visualizing the calculation results for pressure and gas mass flow rate.

Let us present the results of visualization in the Excel environment, obtained for $ct/l = 3$.

Fig. 4 shows the isolines of the gas velocity in the coordinate plane $(ct/l, x)$. The upper left part of the figure corresponds to the zone at rest. In the blue stripes, the velocity is less than 2 m/s. The main changes in velocity occur in the inlet section at the maximum values of the inlet gas velocity. The result is of periodic pattern.

The same considerations can be made regarding the mass flow rate of gas (Fig. 5).

The main changes in pressure occur near the inlet section at the maximum values of the inlet gas velocity (Fig. 6). The highest pressure value is reached at the beginning of the process. A significant change, relative to the given isolines, is a non-uniform directional pressure decrease, especially in

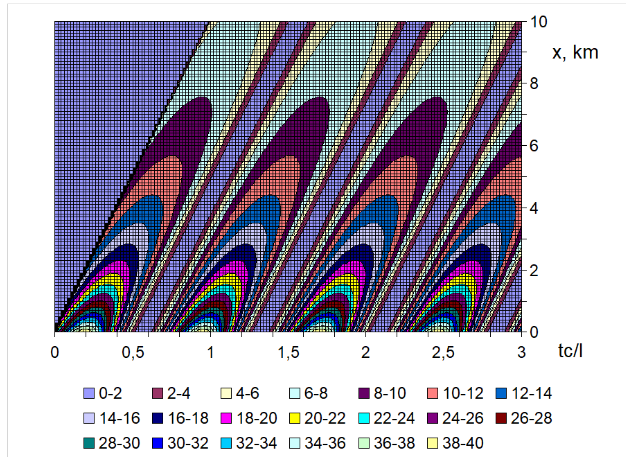

Fig. 5: Isolines of the average gas velocity in plane (x, ct/l) for $u(0,t) = 2\sin 0.1\pi tm/s$.

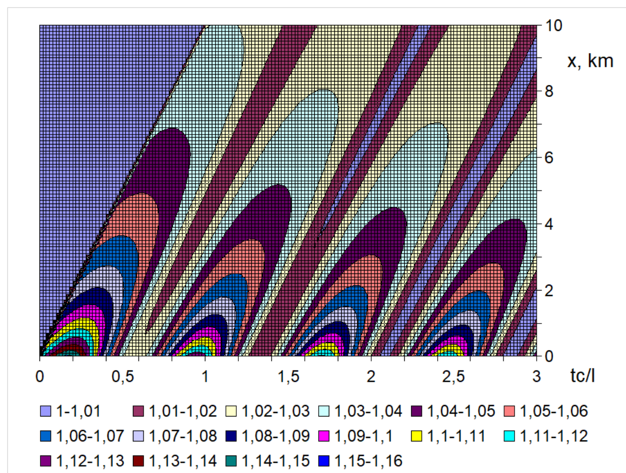

Fig. 6: Isolines of the average gas velocity in plane (x, ct/l) for $u(0,t) = 2\sin 0.1\pi tm/s$.

the first period, where the pressure curves turned out to be asymmetric.

V Analysis and Conclusions

The paper proposes a mixed method for studying the wave motion of gas in a semi-infinite pipeline. In contrast to the known problems, the propagation of a one-way wave is considered. Therefore, the conditions for the unambiguity of the solution to the gas velocity problem are taken in the form of one boundary and two initial conditions.

With the introduction of an auxiliary function in the form of the natural logarithm of the reduced density and with the use of gage functions, the equations are presented in a simpler form with respect to the reference solution of the problem. The reference solution was refined by reducing it to the Riemann problem, considering the quadratic law of resistance; an exact solution to the problem with respect to the gas flow rate was obtained.

To determine the pressure, proportional to the gas density, a finite-difference method was used. The time step was defined as the fraction of the length step to the velocity of small pressure perturbations. The marching method was used to determine the field of velocity and auxiliary function. Using the gas velocity value, according to the new form of the momentum conservation equation, the pressure values at the front of the pulse jump at the beginning of the process and during the pulse propagation were determined. Further, the pressure distribution in the computational domain was determined by recurrent calculation against the direction of an impulse propagation.

The check showed that the proposed mixed analytical-numerical method could be applied to study some problems of the pulse wave propagation in gas pipelines.

One of these problems is the problem of starting up a gas pipeline with a constant gas velocity at the inlet. It was shown that in the perturbed zone the gas velocity decreases by the hyperbolic law. At the front of the impulse, the gas velocity decreases by a jump to a state of rest. A jump-like increase in pressure at the inlet to the gas pipeline was revealed, the cause of which is an abrupt increase in velocity at the beginning of the process. Subsequently, the inlet pressure drops since the gas mass involved to the motion increases. It was found that in the chosen system of discrete coordinates the curves of the isobars and the jump front intersect at a right angle.

The problem of the propagation of a sinusoidal non-negative gas perturbation in a gas medium at rest was considered. In the perturbed zone, the gas velocity is of a periodic pattern. A temporary decrease is observed in the pressure curves in the perturbed zone.

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