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THE THERMODYNAMIC FORMALISM FOR CIRCLE MAPS WITH ALGEBRAIC ROTATION NUMBER

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Abstract– In present paper we study the orientation preserving circle homeomorphisms with singularity of break type. Let $T \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$, $\varepsilon > 0$, be a circle homeomorphism with one break point x_b, at which $T'(x)$ has a discontinuity of the first kind and both one-sided derivatives at the point x_b are strictly positive. Assume that the rotation number ρ_T is irrational and its decomposition into a continued fraction has a form $\rho := \omega_k = [k, k, \ldots, k, \ldots] = \frac{-k+\sqrt{k^2+4}}{2}$, $k \geq 1$. E. Vul and K. Khanin in **?** showed that the renormalization transformation on the space of such circle maps has unique periodic point (F_i, G_i), $i = 1,2$ with period two. Moreover, F_i and $G_i)$ are fractional linear maps. We denote by T_i, $i = 1,2$ the circle homeomorphisms associated by pair (F_i, G_i). Let $B(T_i)$, $i = 1,2$ the set of all circle maps which are C^1 conjugated to T_i, $i = 1,2$. We build a thermodynamic formalism for all maps of $B(T_i)$, $i = 1,2$.

Key words– circle homeomorphism, break point, rotation number, invariant measure, symbolic dynamics, shift map, thermodynamic formalism, renormalization transformation.

I Introduction

This paper is devoted to the construction of a potential for homeomorphisms of a circle with one break point and the number of rotations equal to $\rho = [k, k, \ldots, k, \ldots]$, $k \geq 1$. In the theory of dynamical systems the thermodynamic formalism was introduced by Ya. G. Sinai **?**. Later, thermodynamic formalism was developed in the works of D. Ruelle **?**, R. Bowen **?**, and others.

E. B. Vul, Ya. G. Sinai and K. M. Khanin in **?**, the thermodynamic formalism was used to study an important object of the theory of universality — the Feigenbaum map.

Piecewise-smooth homeomorphisms of the circle is one of the intensively studied fields in the modern theory of dynamical systems. Such maps are a natural generalization of circle diffeomorphisms, as well as an important part of the class of generalized (nonlinear) rearrangements (see **????**). The other hand such maps are interval exchange maps.

It is well known that every orientation-preserving homeomorphism of the circle T with irrational rotation number $\rho = \rho_T$ is strictly ergodic, that is, it has a unique probability T -invariant measure $\mu = \mu_T$ **?**. Let the rotation number $\rho = \rho_T$ is irrational. A. Denjoy showed (see **?**) that if $T \in C^1(S^1)$ is circle diffeomorphism with finite $Var(\log T')$ and irrational rotation number then it is topologically conjugated by linear rotation $T_\rho x = x + \rho, \mod 1$ i.e. there exists a homeomorphism Φ such that $\Phi \circ T = T_\rho \circ \Phi$.

The question of the smoothness of the conjugation Φ and the problem of the absolute continuity of the invariant measure μ_T are closely related. Indeed, an invariant measure μ_T is absolutely continuous with respect to the Lebesgue measure if and only if $\Phi(x)$ is an absolutely continuous function. This reasoning was first used by V. I. Arnold **?**, where he studied the smoothness of $\Phi(x)$. By now, this problem has been completely solved in a certain sense for diffeomorphisms of the circle. It is well known that for sufficiently smooth maps T with a typical irrational number $\rho = \rho_T$ the unique invariant measure μ_T is absolutely continuous with respect to Lebesgue measure (see **????**).

Piecewise smooth homeomorphisms with breaks are a natural generalization of circle diffeomorphisms. The simplest examples of piecewise smooth maps are piecewise linear (PL) homeomorphisms with two breaks. First, such circle maps were studied by M. Herman **?**. M. Herman proved **?** that the invariant PL measure of the homeomorphism h with two breaks and an irrational rotation number is absolutely continuous if and only if both break points lie on the same orbit. For homeomorphisms of a circle with one break point, the character of the invariant measure is very different to the case of diffeomorphisms. A. Dzhalilov and K. Khanin in **?** proved that for a circle homeomorphism $T \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$, $\varepsilon > 0$, with one break point x_b and irrational rotation number ρ_T the invariant measure μ_T is singular with respect to the Lebesgue measure λ, that is, there is a measurable subset $A \subset S^1$ such that $\mu_T(A) = 1$ and $\lambda(A) = 0$.

Consider two homeomorphisms T_1 and T_2 with the same irrational rotation number $\rho = \rho(T_1) = \rho(T_2)$ and with one breakpoint $x_0 = x_b$. The question of the regularity of the conjugation Φ between T_1 and T_2 is called the "rigidity" problem. This problem was intensively studied during last 15-20 years by many authors (see for instance **??**).

II Preliminaries

In this section, we will consider all the concepts necessary for formulating the theorem on thermodynamic formalism, including the renormalization group transformation in the space of homeomorphisms of a circle with one breakpoint and with a rotation number ρ (for more details see **??**), dynamic circle partitioning (see **?** for more details), and symbolic dynamics (see **?**).

The renormalization group transformation in the space of homeomorphisms of a circle with breaks and an algebraic rotation number has a periodic orbit **?**. We will construct a potential for a periodic trajectory with a rotation number ρ. We denote by X_b the set of pairs of strictly increasing functions ($f(x)$, $x \in [-1,0]$, $g(x)$, $x \in [0,\alpha]$) satisfying the following conditions:

a) $f(0) = \alpha > 0$, $g(0) = -1$;

b) $f(-1) = g(\alpha)$;

c) $f(g(0)) = f(-1) < 0$;

d) $f^{(2)}(g(0)) \geq 0$;

e) $f(x) \in C^{2+\varepsilon}([-1,0])$, $g(x) \in C^{2+\varepsilon}([0,\alpha])$ for any $\varepsilon > 0$.

Conditions a) - c) allow using $(f,g) \in X_b$ to construct a homeomorphism of the circle $[-1,\alpha)$ by the formula:

$$T_{f,g}(x) = \begin{cases} f(x), & \text{if } x \in [-1,0), \\ g(x), & \text{if } x \in [0,\alpha). \end{cases}$$

Homeomorphism $T_{f,g}(x)$ by linear map $l(x) = \frac{x+1}{\alpha+1}$ becomes a homeomorphism of the circle $S^1 = [0,1)$. The rotation number $T_{f,g}(x)$ is defined as the rotation number of the homeomorphism $l \circ T_{f,g} \circ l^{-1}$.

We denote by $X_b(\omega)$ the subset consisting of pairs $(f,g) \in X_b$ such that the rotation number $\rho(T_{f,g}) := \omega = [k,k,\ldots,k,\ldots] = \frac{-k+\sqrt{k^2+4}}{2}$, $k \geq 1$.

We define the transformation of the renormalization group $R_b\colon X_b(\omega) \to X_b(\omega)$ by the formula (see **??**):

$$R_b(f(x),g(x)) = (\tilde{f}(x), \quad x \in [-1,0]; \qquad \tilde{g}(x), \quad x \in [0,\alpha']),$$

where

$$\tilde{f}(x) = -\alpha^{-1} f(g(-\alpha x)), \qquad \tilde{g}(x) = -\alpha^{-1} f(-\alpha x),$$

$$\alpha' = -\alpha^{-1} f(-1).$$

Note that the $(\tilde{f},\tilde{g})$ pair corresponds to the map of the first return in new linear coordinates.

Determine the jump value of the break at the point x=0:

$$c = \sqrt{\frac{f'(-0)}{f'(+0)}}.$$

It is clear that for $c = 1$ we get a smooth map. In what follows, we will assume that $c \neq 1$. In the paper **?** it is proved that for a fixed c the transformation R_b in the subset $X_b(\omega)$ has a unique periodic trajectory

$$\{f_i(x,c_i), g_i(x,c_i), i = 1,2\}$$

of period two. It means that

$$R_b(f_1(x,c_1), g_1(x,c_1)) = (f_2(x,c_2), g_2(x,c_2)),$$

$$R_b(f_2(x,c_2), g_2(x,c_2)) = (f_1(x,c_1), g_1(x,c_1)).$$

Functions $f_i(x,c_i)$ and $g_i(x,c_i)$, $i = 1,2$, have the following form:

$$f_i(x,c_i) = \frac{(\alpha_i + c_i x)\beta_i}{\beta_i + (\beta_i + \alpha_i - c_i)x}, \tag{II.1}$$

$$g_i(x,c_i) = \frac{\alpha_i\beta_i(x_i - c_i)}{\alpha_i\beta_i c_i + (c_i - \alpha_i - c_i\beta_i)x}, \tag{II.2}$$

where

$$\alpha_1 = \frac{c - \beta_0^2}{1+\beta_0}, \qquad \alpha_2 = \frac{c^{-1} - \beta_0^2}{1+\beta_0}, \qquad c_1 = c, \qquad c_2 = c^{-1},$$

$$\beta_1 = \beta_2 = \beta_0,$$

β_0 — unique root of equation

$$\beta^4 - \beta^3 - \beta^2\frac{(c+1)^2}{c} - \beta + 1 = 0,$$

belonging to the interval $(0,1)$.

Identifying the ends of the intervals $[-1,\alpha_i)$, $i = 1,2$, we obtain the circles S_i, $i = 1,2$. Now, using (f_i,g_i), $i = 1,2$, we define homeomorphisms of the circle $T_i\colon S_i \to S_i$ by the formula:

$$T_i(x) = \begin{cases} f_i(x,c_i), & \text{if } x \in [-1,0), \\ g_i(x,c_i), & \text{if } x \in [0,\alpha_i). \end{cases}$$

Below we describe the properties of the homeomorphism T_1 of the circle S_1. The homeomorphism T_1 has breaks at the points x_0 and $x_1 = T_1(x_0)$, and the product of the magnitudes of the breaks at these points is c_1. We denote the homeomorphism T_1 by T_b. The second homeomorphism T_2 is studied in a similar way.

We denote by $B(T_b)$ the set of all $C^{1+\theta}$ - conjugate homeomorphisms of T_b. In this section, we will construct a thermodynamic formalism for maps belonging to $B(T_b)$. Take an arbitrary homeomorphism $T \in B(T_b)$. The map $T \in C^{2+\varepsilon}(S^1 \setminus \{x_0, T(x_0)\})$, $\varepsilon > 0$, has two breakpoints x_0 and $T(x_0)$, and the rotation number is ω.

We denote by $\frac{p_n}{q_n}$, $n \geq 1$, the n -th fraction of ω. The numbers q_n satisfy the following difference equation: $q_{n+1} = q_n + q_{n-1}$, $q_0 = 1$, $q_1 = 1$. The q_n numbers are called Fibonacci numbers. Take an arbitrary point $x_0 \in S^1$ and consider its orbit

$$\mathbb{O}_T(x_0) = \{x_0, x_1 = T(x_0), x_2 = T^2(x_0), \ldots, x_n = T^n(x_0), \ldots\}.$$

Here and in what follows, T^n denotes the n -th iteration of T. Using the orbit $\mathbb{O}_T(x_0)$, we define the sequence $\{\mathbb{P}_n(x_0), n \geq 1\}$ dynamic partitions of the circle.

The partition $\mathbb{P}_n(x_0)$ is obtained using a part of the orbit of the point x_0: $\{x_i,\ 0 \leq q_n + q_{n+1} - 1\}$. For each $n \geq 1$, we denote by $\Delta_0^{(n)}(x_0)$ the segment connecting the points x_0 and x_{q_n}.

Let $\Delta_i^{(n)}(x_0) = T^i(\Delta_0^{(n)}(x_0))$, $i \geq 0$. Then the partition $\mathbb{P}_n(x_0)$ consists of the system of segments $\{\Delta_i^{(n)},\ 0 \leq i < q_{n+1}\}$ and $\{\Delta_j^{(n+1)},\ 0 \leq j < q_n\}$ (see **?**). The partition $\mathbb{P}_n(x_0)$ is called the *n - th dynamic partition* of the circle. Note that any two segments of the partition $\mathbb{P}_n(x_0)$ can intersect only by endpoints. When moving from $\mathbb{P}_n(x_0)$ to $\mathbb{P}_{n+1}(x_0)$ all "short" segments $\Delta_j^{(n+1)}(x_0)$, $0 \leq j \leq q_n - 1$, are preserved, and the "long" segments $\Delta_i^{(n)}(x_0)$, $0 \leq i < q_{n+1}$, is divided into pairs of segments:

$$\Delta_i^{(n)} = \Delta_i^{(n+2)} \bigcup \Delta_{i+q_n}^{(n+1)}. \tag{II.3}$$

Using the sequence of dynamic partitions $\mathbb{P}_n(x_0)$, one can construct a symbolic dynamics as follows. Let $x \in S^1 \backslash \mathbb{O}_T(x_0)$. Suppose $a_{n+1} := a_{n+1}(x) = a$, if $x \in \Delta_i^{(n+1)}(x_0)$, $0 \leq i < q_n$. Let $x \in \Delta_i^{(n)}(x_0)$, $0 \leq i < q_{n+1}$. Due to (II.3), the point x falls into the segment $\Delta_i^{(n+2)}(x_0)$ or into the segment $\Delta_{i+q_n}^{(n+1)}(x_0)$. We put in the first case $a_{n+1} = 0$, and in the second $a_{n+1} = 1$. Thus, we get a one-to-one correspondence

$$\varphi\colon S^1 \backslash O_f(x_0) \leftrightarrow \{\underline{a} = (a_1, a_2, \ldots a_n, \ldots), \quad a_i \in \{a, 0, 1\},$$
$$\text{wherein } a_{n+1} = a \text{ if and only if } a_n = 0,\ n \geq 1\} =: \Theta_+.$$

Note that in this case, each segment $\Delta^{(n)}$ of the dynamic partition $\mathbb{P}_n(x_0)$ corresponds to a unique word of length n: $(a_1, a_2, \ldots, a_n)$. In particular, the words $(0, a, 0, a, \ldots, 0, a)$ and $(a, 0, a, 0, \ldots, a, 0)$ correspond to the segments $\Delta_0^{(n)}$ and $\Delta_0^{(n+1)}$ respectively. Let $\Delta^{(n)} := \Delta(a_1, a_2, \ldots, a_n)$. The Lebesgue measure on S^1 induces the probability measure λ_0 on Θ_+:

$$\lambda_0(a_1, a_2, \ldots, a_n) := |\Delta(a_1, a_2, \ldots, a_n)|.$$

When passing from the circle S^1 to the space of infinite words Θ_+, the map T goes in $\widetilde{T}\colon \Theta_+ \to \Theta_+$. Using the structure of dynamic partitions, one can easily verify that $\widetilde{T}$ is not a Bernoulli shift.

Now we define another space Ω of one-sided infinite words with the same alphabet $a, 0, 1$:

$$\Omega := \{\underline{a} = (a_1, a_2, \ldots, a_n, \ldots), \quad a_i \in \{a, 0, 1\},$$
$$\text{wherein } a_{n+1} = 0 \text{ if and only if } a_n = a,\ n \geq 1\}.$$

In what follows, by $\vec{a}$ we will denote the vector $(a_1, a_2, \ldots, a_n)$, and by $\underline{b}$ we will denote the infinite word $(b_1, b_2, \ldots, b_n, \ldots)$.

We define the following function

$$\underline{\gamma}(x) = \begin{cases} (a, 0, a, 0, \ldots, a, 0, \ldots), & \text{if } x = a, \\ (0, a, 0, a, \ldots, 0, a, \ldots), & \text{if } x = 0, 1. \end{cases}$$

III A THEOREM ON THERMODYNAMIC FORMALISM FOR HOMEOMORPHISMS OF A CIRCLE WITH BREAKS

Let us now formulate the main result of this paper, the theorem on thermodynamic formalism.

Theorem III.1 *For all maps $T \in B(T_b)$ with one break point and an irrational number of rotations $\rho := \omega_k = [k, k, \ldots, k, \ldots] = \frac{-k+\sqrt{k^2+4}}{2}$, $k \geq 1$, there exists a unique continuous (in the Tikhonov topology) function $U_b : \Omega \to (-\infty, 0)$ with the following properties:*

1) For any $\underline{a} = (a_1, \ldots, a_k, a_{k+1}, \ldots, a_n, \ldots)$ and $\underline{b} = (a_1, \ldots, a_k, b_{k+1}, \ldots, b_n, \ldots)$ the space Ω satisfies the estimate

$$|U_b(\underline{a}) - U_b(\underline{b})| \leq const \cdot q^k,$$

where constant $q \in (0,1)$ - does not depend on $\underline{a}$, $\underline{b}$ and k.

2) Let $\Delta_{s_n}^{(n)} \subset \Delta_{s_r}^{(r)}$, $1 \leq r < n$ and $\varphi(\Delta_{s_n}^{(n)}) = (b_1, \ldots, b_n)$, $\varphi(\Delta_{s_r}^{(r)}) = (b_1, \ldots, b_r)$, then

$$|\Delta_{s_n}^{(n)}| = (1 + \psi(b_1, b_2, \ldots, b_n))|\Delta_{s_r}^{(r)}| \times$$
$$\times \exp\{\sum_{s=r}^{n} U_b(b_s, b_{s-1}, \ldots, b_r, \ldots, b_1, \underline{\gamma}(b_1))\},$$

where $|\psi(b_1, b_2, \ldots, b_n)| \leq Const \cdot q^r$.

A similar result for homeomorphisms of a circle with one break point and an irrational number of rotations equal to the golden ratio, i.e. $\rho = [1,1,1...,1,...] = \frac{\sqrt{5}-1}{2}$, was proved in work **?**.

Note that the proof of Theorem III.1 essentially uses the method of thermodynamic formalism.

The second statement of Theorem III.1 implies that the potential U_b is uniquely determined as the limit of the ratio of the lengths of segments of dynamic partitions $\mathbb{P}_n$ breakpoints x_0 of the map T. In other words, the dynamics of the singular point x_0 uniquely determines the potential corresponding to T, therefore, only one potential U_b corresponds to the map T.

References