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COMPLETE CLASSIFICATION OF TWO-DIMENSIONAL ALGEBRAS OVER THE FIELD OF RATIONAL NUMBERS

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Abstract– A full classification,up to isomorphism, of two-dimensional algebras over the field of rational numbers is presented. The provided in this paper method can be successfully used to classify two-dimensional algebras over the quotient field of any factorial ring as well.

Key words– algebra, group, homomorphism, isomorphism, matrix of structure constants.

I Introduction

The classification of mathematical structures, up to the corresponding isomorphisms, is considered as one of the important problems. In this paper we consider classification, up to isomorphism, of two-dimensional algebras over the field of rational numbers. Our approach to this problem is a coordinate (basis, structure constants) based approach. One advantage of this approach is that when a full classification is obtained then many problems related to these algebras can be handled quite easily. In two-dimensional case a complete classification, by basis free approach, is stated in [1] over any basic field. So far its application to solve many problems related to two-dimensional algebras is not known and probably one should not expect big help from that basis free approach. In opposite a coordinate based complete classification of two-dimensional algebras over any field, where any second and third order polynomials has root, and over the field of real numbers is given in [2]. In [3-7] one can find their applications to different problems related to such algebras. Another coordinate based approach to this problem and its applications one can find in [8]. It should be noted that classification of all n-dimensional algebras over a field, usually assumed algebraically closed, is obtained only in $n=2$ case. In $n>2$ case there are many investigations on classification of special, usually polynomial-identity, classes of algebras.

To get our main result we follow the same approach used in [2] but whenever in process of proof we run into a second or third order polynomial which has no rational root we manage to find a way out.

II The main part

Further we need the following result which is given in [9] in a slightly different form. Let n be a natural number, G be a group, $\mathbb{F}$ be any field, $\tau: G\to GL(n,\mathbb{F})$ be a homomorphism of groups and $M\subset \mathbb{F}^n$ be a τ-invariant subset, that is $\tau(g)u\in M$ whenever $u\in M$ and $g\in G$. We write $u\simeq^G v$ if $\tau(g)u=v$ for some $g\in G$.

Lemma. *If there exist $m\in\mathbb{N}$, a map $P: M\to GL(m,\mathbb{F})$ and a homomorphism $\tau': G'\to GL(n,\mathbb{F})$ such that*

$$\tau'(P(\tau(g)u))=\tau'(P(u))\tau(g^{-1}) \ \ whenever \ \ u\in M,\ g\in G$$

and $v,w\in M$ are any two elements then $v\simeq^G w$ if and only if

$$\tau'(P(v))v=\tau'(P(w))w \ \ and \ \ \tau'(P(w)^{-1}P(v))\in\tau(G),$$

where G' stands for the subgroup of $GL(m,\mathbb{F})$ generated by $\{P(u): u\in M\}$.

Let $\mathbf{A}$ be any 2 dimensional algebra over $\mathbb{Q}$ with multiplication $\cdot$ given by a bilinear map $(\mathbf{u},\mathbf{v})\mapsto \mathbf{u}\cdot\mathbf{v}$ whenever $\mathbf{u},\mathbf{v}\in\mathbf{A}$. If $e=(e^1,e^2)$ is a basis for $\mathbf{A}$ as a vector space over $\mathbb{Q}$ then one can represent this bilinear map by a matrix

$$A=\left(\begin{array}{cccc} A_{1,1}^1 & A_{1,2}^1 & A_{2,1}^1 & A_{2,2}^1\\ A_{1,1}^2 & A_{1,2}^2 & A_{2,1}^2 & A_{2,2}^2\end{array}\right)\in Mat(2\times 4;\mathbb{Q})$$

such that

$$\mathbf{u}\cdot\mathbf{v}=eA(u\otimes v)$$

for any $\mathbf{u}=eu, \mathbf{v}=ev$, where $u=(u_1,u_2)$, and $v=(v_1,v_2)$ are column coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively, $(u\otimes v)=(u_1v_1,u_1v_2,u_2v_1,u_2v_2)$, $e^i\cdot e^j=A^1_{i,j}e^1+A^2_{i,j}e^2$ whenever $i,j=1,2$. So the algebra $\mathbf{A}$ is presented by the matrix $A\in Mat(2\times 4;\mathbb{Q})$, called the matrix of structure constants (MSC) of $\mathbf{A}$ with respect to the basis e.

If $e'=(e'^1,e'^2)$ is also a basis for $\mathbf{A}$, $g\in G=GL(2,\mathbb{Q})$, $e'g=e$ and $\mathbf{u}\cdot\mathbf{v}=e'B(u'\otimes v')$, where $\mathbf{u}=e'u', \mathbf{v}=e'v'$, then

$$\mathbf{u}\cdot\mathbf{v}=eA(u\otimes v)=e'B(u'\otimes v')=eg^{-1}B(g\otimes g)(u\otimes v)$$

as far as $\mathbf{u}=eu=e'u'=eg^{-1}u', \mathbf{v}=ev=e'v'=eg^{-1}v'$. Therefore the equality

$$B=gA(g^{-1})^{\otimes 2}$$

is valid, where for $g^{-1}=\begin{pmatrix}\xi_1 & \eta_1\\ \xi_2 & \eta_2\end{pmatrix}$ one has

$$(g^{-1})^{\otimes 2}=g^{-1}\otimes g^{-1}=\begin{pmatrix}\xi_1^2 & \xi_1\eta_1 & \xi_1\eta_1 & \eta_1^2\\ \xi_1\xi_2 & \xi_1\eta_2 & \xi_2\eta_1 & \eta_1\eta_2\\ \xi_1\xi_2 & \xi_2\eta_1 & \xi_1\eta_2 & \eta_1\eta_2\\ \xi_2^2 & \xi_2\eta_2 & \xi_2\eta_2 & \eta_2^2\end{pmatrix}.$$

Definition. *Two-dimensional algebras* $\mathbf{A}$, $\mathbf{B}$, *given by their matrices of structural constants A, B, are said to be isomorphic if* $B=gA(g^{-1})^{\otimes 2}$ *holds true for some* $g\in GL(2,\mathbb{Q})$.

Note that the following identities

$$Tr_1(gA(g^{-1})^{\otimes 2})=Tr_1(A)g^{-1},\ Tr_2(gA(g^{-1})^{\otimes 2})=Tr_2(A)g^{-1} \quad (1)$$

hold true whenever $A\in Mat(2\times 4,\mathbb{Q}), g\in GL(2,\mathbb{Q})$, where

$$Tr_1(A)=(A^1_{1,1}+A^2_{2,1},A^1_{1,2}+A^2_{2,2}),$$

$\mathrm{Tr}_2(A)=(A^1_{1,1}+A^2_{1,2},A^1_{2,1}+A^2_{2,2})$ are row vectors.

To prove our main result we divide $Mat(2\times 4,\mathbb{Q})$ into the following five disjoint subsets:

1. All A for which the system $\{Tr_1(A),Tr_2(A)\}$ is linear independent.

2. All A for which the system $\{Tr_1(A),Tr_2(A)\}$ is linear dependent and $Tr_1(A),Tr_2(A)$ are nonzero vectors.

3. All A for which $Tr_1(A)$ is nonzero vector and $Tr_2(A)=(0,0)$.

4. All A for which $Tr_1(A)=(0,0)$ and $Tr_2(A)$ is nonzero vector.

5. All A for which$Tr_1(A)=Tr_2(A)=(0,0)$.

Due to equalities (1) it is clear that algebras with MSC from these different classes can not be isomorphic. We deal with each of these subsets separately. Further, for the simplicity, we use the notation

$$A=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\end{pmatrix},$$

where variables $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4$ stand for any elements of $\mathbb{Q}$. We use the same variables for the entries of MSC to avoid too many variables. Here is the main result

Theorem. *Any non-trivial 2-dimensional algebra over the field of rational numbers* $\mathbb{Q}$ *is isomorphic to only one of the following listed, by their matrices of structure constants, algebras:*

$$A_1(\mathbf{c})=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4\\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2\end{pmatrix}, \textit{ where}$$

$\mathbf{c}=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{Q}^4$,

$$A_2(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & p\\ \beta_1 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}, \textit{ where}$$

$\mathbf{c}=(\alpha_1,p,\beta_1,\beta_2)\in\mathbb{Q}^3,\ \beta_1\geq 0, p\in\mathbb{Z}$*-without prime square divisor(square-free integer),*

$$A_3(\mathbf{c})=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & \beta_2 & 1 & -1\end{pmatrix}, \textit{ where } \mathbf{c}=(\beta_1,\beta_2)\in\mathbb{Q}^2,$$

$$A_4(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}, \textit{ where } \mathbf{c}=(\alpha_1,\beta_2)\in\mathbb{Q}^2,$$

$$A_5(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix}, \textit{ where } \mathbf{c}=\alpha_1\in\mathbb{Q},$$

$$A_6(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & p\\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}, \textit{ where}$$

$\mathbf{c}=(\alpha_1,p,\beta_1)\in\mathbb{Q}^3,\ \beta_1\geq 0, p\in\mathbb{Z}$*-without prime square divisor,*

$$A_7(\beta_1)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 1 & 0 & -1\end{pmatrix}, \textit{ where } \beta_1\in\mathbb{Q},$$

$$A_8(\alpha_1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}, \textit{ where } \alpha_1\in\mathbb{Q},$$

$$A_9=\begin{pmatrix}\frac{1}{3} & 0 & 0 & 0\\ 1 & \frac{2}{3} & -\frac{1}{3} & 0\end{pmatrix},$$

$$A_{10}(\mathbf{c})=\begin{pmatrix}0 & 1 & 1 & 1\\ \beta_1 & 0 & 0 & -1\end{pmatrix}\simeq\begin{pmatrix}0 & 1 & 1 & 1\\ \beta_1'(a) & 0 & 0 & -1\end{pmatrix},$$

where polynomial $(\beta_1 t^3 - 3t - 1)(\beta_1 t^2 + \beta_1 t + 1)$ *has no rational root,* $\frac{-1}{2} \neq a \in \mathbb{Q}$ *and* $\beta_1'(t) = \frac{(\beta_1^2 t^3 + 6\beta_1 t^2 + 3\beta_1 t + \beta_1 - 2)^2}{(\beta_1 t^2 + \beta_1 t + 1)^3}$,

$$A_{11}(p) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ p & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ \overline{p} & 0 & 0 & 0 \end{pmatrix},$$

where $0 < p \in \mathbb{Z}, 0 < \overline{p} \in \mathbb{Z}$ *are not divisible by cube of any prime number,* $p\overline{p} = q^3$ *for some* $q \in \mathbb{Z}$.

$$A_{12}(p) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ p & 0 & 0 & -1 \end{pmatrix}, \text{ where } p = 0 \text{ or } p \in \mathbb{Z} \setminus \{0\}$$

-without prime square divisor,

$$A_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. The first subset case. In this case let $P(A)$ stand for the nonsingular matrix $\begin{pmatrix} \alpha_1 + \beta_3 & \alpha_2 + \beta_4 \\ \alpha_1 + \beta_2 & \alpha_3 + \beta_4 \end{pmatrix}$ with rows

$$Tr_1(A) = (\alpha_1 + \beta_3, \alpha_2 + \beta_4),\ Tr_2(A) = (\alpha_1 + \beta_2, \alpha_3 + \beta_4).$$

Due to (1) we have the identity

$$P(gA(g^{-1})^{\otimes 2}) = P(A)g^{-1}.$$

Therefore according to Lemma two-dimensional algebras **A**, **B**, given by their matrices of structure constants A, $B \in M$, are isomorphic if and only if the equality

$$P(B)B(P(B)^{-1} \otimes P(B)^{-1}) = P(A)A(P(A)^{-1} \otimes P(A)^{-1})$$

holds true, where

$$M = \{A \in Mat(2 \times 4, \mathbb{Q}) : \det(P(A)) \neq 0.\}$$

It is easy to see that in this case

$$P(P(A)A(P(A)^{-1} \otimes P(A)^{-1})) = I_2$$

-the second order identity matrix. Therefore, using obvious re-notations, one can list the following non-isomorphic "canonical" representatives of algebras from the first subset, given by their MSCs as follows:

$$A_1(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 + 1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1 + 1 & -\alpha_2 \end{pmatrix}, \text{ where}$$

$\mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{Q}^4$.

The second and third subset cases. In these cases one can make $Tr_1(A)g = (1,0)$ and therefore $Tr_2(A)g = (\lambda, 0)$ for some $\lambda \in \mathbb{Q}$. It implies that one can consider

$$Tr_1(A) = (\alpha_1 + \beta_3, \alpha_2 + \beta_4) = (1,0) \text{ and}$$

$$Tr_2(A) = (\alpha_1 + \beta_2, \alpha_3 + \beta_4) = (\lambda, 0).$$

Here $\lambda = 0$ case covers the third subset case. So let us consider

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 & \alpha_4 \\ \beta_1 & \lambda - \alpha_1 & 1 - \alpha_1 & -\alpha_2 \end{pmatrix},$$

with respect to $g \in GL(2, \mathbb{Q})$ of the form

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ \xi_2 & \eta_2 \end{pmatrix},$$

as far as $(1,0)g^{-1} = (1,0)$ if and only if g^{-1} is of the above form. In this case for the entries of $A' = \begin{pmatrix} \alpha_1' & \alpha_2' & \alpha_2' & \alpha_4' \\ \beta_1' & \lambda - \alpha_1' & 1 - \alpha_1' & -\alpha_2' \end{pmatrix} = gA(g^{-1})^{\otimes 2}$, as far as λ is same for the A and A', one has

$\alpha_1' = \alpha_1 + 2\alpha_2\xi_2 + \alpha_4\xi_2^2$,
$\alpha_2' = (\alpha_2 + \alpha_4\xi_2)\eta_2$,
$\alpha_4' = \alpha_4\eta_2^2$,
$\beta_1' = \frac{\beta_1 + (1 + \lambda - 3\alpha_1)\xi_2 - 3\alpha_2\xi_2^2 - \alpha_4\xi_2^3}{\eta_2}$.

Now we have to consider a few cases.

Case 1: Let $\alpha_4 = \frac{m}{n} \neq 0$, where $n, m \in \mathbb{Z}$, be in lowest terms. One can represent $\frac{m}{n}$ in the following unique form $\frac{m}{n} = p(\frac{m_1}{n_1})^2$, where p is a nonzero integer not divisible by square of any prime number. Therefore by taking $\eta_2 = \pm\frac{n_1}{m_1}$ and appropriate ξ_2 one can make $\alpha_2' = 0, \alpha_4' = p$ to come to the "canonical" MSCs as follows

$$A_2(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & p \\ \beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & p \\ -\beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix},$$

where $\mathbf{c} = (\alpha_1, p, \beta_1, \beta_2) \in \mathbb{Q}^4$, $p \in \mathbb{Z}$- without prime square divisor.

Case 2: $\alpha_4 = 0$.

Subcase 2 - a): $\alpha_2 \neq 0$. If $\alpha_2 \neq 0$ then one can make $\alpha_1' = 0$, $\alpha_2' = 1$, $\beta_1' = \alpha_2\beta_1 - \frac{2 + 2\lambda - 3\alpha_1}{4}\alpha_1$ to get the following set of canonical matrices of structural constants

$$A_3(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}, \text{ where } \mathbf{c} = (\beta_1, \beta_2) \in \mathbb{Q}^2.$$

Subcase 2 - b): $\alpha_2 = 0$. If $\alpha_2 = 0$ then $\alpha_1' = \alpha_1$, $\alpha_2' = 0$, $\alpha_4' = 0$, $\beta_1' = \frac{\beta_1 + (1 + \lambda - 3\alpha_1)\xi_2}{\eta_2}$.

Subsubcase: 2 - b) - 1: $1 + \lambda - 3\alpha_1 \neq 0$. If $1 + \lambda - 3\alpha_1 \neq 0$, that is $\lambda - \alpha_1 \neq 2\alpha_1 - 1$, one can make $\beta_1' = 0$ to get

$$A_4(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix}, \text{ where } \mathbf{c} = (\alpha_1, \beta_2) \in \mathbb{Q}^2,$$

with $\beta_2 \neq 2\alpha_1 - 1$.

Subsubcase: 2 - b) - 2: $1+\lambda-3\alpha_1=0$. If $1+\lambda-3\alpha_1=0$ and $\beta_1\neq 0$ one can make $\beta_1'=1$ to get

$$A_5(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix},\text{ where } \mathbf{c}=\alpha_1\in\mathbb{Q}.$$

If $\beta_1=0$ one has $\lambda-\alpha_1=2\alpha_1-1$ and therefore

$$A'=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix}$$

which is $A_4(\mathbf{c})$ with $\beta_2=2\alpha_1-1$.

The fourth subset case. By the similar justification as in the second and the third subsets case it is enough to consider

$$A=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_2 & \alpha_4\\ \beta_1 & 1-\alpha_1 & -\alpha_1 & -\alpha_2\end{pmatrix}\text{ and } g^{-1}=\begin{pmatrix}1 & 0\\ \xi_2 & \eta_2\end{pmatrix},$$

where one has

$\alpha_1'=\alpha_1+2\alpha_2\xi_2+\alpha_4\xi_2^2,$
$\alpha_2'=(\alpha_2+\alpha_4\xi_2)\eta_2,$
$\alpha_4'=\alpha_4\eta_2^2,$
$\beta_1'=\frac{\beta_1+\xi_2-3\alpha_1\xi_2-3\alpha_2\xi_2^2-\alpha_4\xi_2^3}{\eta_2}.$

Therefore we get the canonical MSCs as follows

$$A_6(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & p\\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}\simeq$$

$$\begin{pmatrix}\alpha_1 & 0 & 0 & p\\ -\beta_1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix},$$

where $\mathbf{c}=(\alpha_1,p,\beta_1)\in\mathbb{Q}^3$, $p\in\mathbb{Z}$- without prime square divisor (it is $A_2(\mathbf{c})$, where the 2^{nd} and 3^{rd} columns are interchanged and $\alpha_1+\beta_2=0$).

$$A_7(\mathbf{c})=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 1 & 0 & -1\end{pmatrix},\text{ where } \mathbf{c}=\beta_1\in\mathbb{Q},$$

(it is $A_3(\mathbf{c})$ with $\beta_2=0$, where the 2^{nd} and 3^{rd} columns are interchanged),

$$A_8(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix},\text{ where } \mathbf{c}=\alpha_1\in\mathbb{Q},$$

(it is $A_4(\mathbf{c})$, where the 2^{nd} and 3^{rd} columns are interchanged and $\alpha_1+\beta_2=0$) or

$$A_9=\begin{pmatrix}\frac{1}{3} & 0 & 0 & 0\\ 1 & \frac{2}{3} & -\frac{1}{3} & 0\end{pmatrix}$$

(it is $A_5(\mathbf{c})$, where the 2^{nd} and 3^{rd} columns are interchanged and $3\alpha_1-1=0$).

The fifth subset case. In this case

$$A=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_2 & \alpha_4\\ \beta_1 & -\alpha_1 & -\alpha_1 & -\alpha_2\end{pmatrix}\text{ and if } g^{-1}=\begin{pmatrix}\xi_1 & \eta_1\\ \xi_2 & \eta_2\end{pmatrix},$$

$\Delta=\xi_1\eta_2-\xi_2\eta_1$ and

$$A'=\begin{pmatrix}\alpha_1' & \alpha_2' & \alpha_2' & \alpha_4'\\ \beta_1' & -\alpha_1' & -\alpha_1' & -\alpha_2'\end{pmatrix}=gA(g^{-1})^{\otimes 2}\text{ then}$$

$\alpha_1'=\frac{1}{\Delta}\left(-\beta_1\eta_1\xi_1^2+\alpha_1\eta_2\xi_1^2+2\alpha_1\eta_1\xi_1\xi_2+2\alpha_2\eta_2\xi_1\xi_2+\alpha_2\eta_1\xi_2^2+\alpha_4\eta_2\xi_2^2,\right.$
$\alpha_2'=\frac{-1}{\Delta}\left(\beta_1\eta_1^2\xi_1-2\alpha_1\eta_1\eta_2\xi_1-\alpha_2\eta_2^2\xi_1-\alpha_1\eta_1^2\xi_2-2\alpha_2\eta_1\eta_2\xi_2-\alpha_4\eta_2^2\xi_2,\right.$
$\alpha_4'=\frac{-1}{\Delta}\left(\beta_1\eta_1^3-3\alpha_1\eta_1^2\eta_2-3\alpha_2\eta_1\eta_2^2-\alpha_4\eta_2^3\right),$
$\beta_1'=\frac{1}{\Delta}\left(\beta_1\xi_1^3-3\alpha_1\xi_1^2\xi_2-3\alpha_2\xi_1\xi_2^2-\alpha_4\xi_2^3\right).$

Let us consider the case when none of the polynomials $\beta_1-3\alpha_1t-3\alpha_2t^2-\alpha_4t^3$, $\beta_1t^3-3\alpha_1t^2-3\alpha_2t-\alpha_4$ has a rational root, which, in particular, implies that $\alpha_4\beta_1\neq 0$. It is easy to see that if $\xi_1=0$ and $\eta_2=-\frac{\alpha_2}{\alpha_4}\eta_1$ then $\alpha_1'=0$. So it is enough to consider only $\alpha_1=0$ case.

Assume that $\alpha_1=0$, $\alpha_2\neq 0$ and consider $\xi_2=\eta_1=0$. In this case due to
$\alpha_1'=0,\ \alpha_2'=\alpha_2\eta_2,\ \alpha_4'=\alpha_4\frac{\eta_2^2}{\xi_1},\ \beta_1'=\beta_1\frac{\xi_1^2}{\eta_2}$ one can make $\alpha_2'=1,\ \alpha_4'=1$. Therefore without lost of generality one can assume that $\alpha_1=0,\ \alpha_2=1,\ \alpha_4=1$ and

$\alpha_1'=\frac{1}{\Delta}\left(-\beta_1\eta_1\xi_1^2+2\eta_2\xi_1\xi_2+\eta_1\xi_2^2+\eta_2\xi_2^2\right),$
$\alpha_2'=\frac{-1}{\Delta}\left(\beta_1\eta_1^2\xi_1-\eta_2^2\xi_1-2\eta_1\eta_2\xi_2-\eta_2^2\xi_2\right),$
$\alpha_4'=\frac{-1}{\Delta}\left(\beta_1\eta_1^3-3\eta_1\eta_2^2-\eta_2^3\right),$
$\beta_1'=\frac{1}{\Delta}\left(\beta_1\xi_1^3-3\xi_1\xi_2^2-\xi_2^3\right).$

In $\xi_2\eta_1=0$ case making $\alpha_1'=0$ results in $g=I_2$. In $\xi_2\eta_1\neq 0$ case $\alpha_1'=0$ if and only if $\frac{\eta_2}{\eta_1}(2\frac{\xi_1}{\xi_2}+1)-\beta_1(\frac{\xi_1}{\xi_2})^2+1=0$ that is $\frac{\eta_2}{\eta_1}=\frac{\beta_1(\frac{\xi_1}{\xi_2})^2-1}{2\frac{\xi_1}{\xi_2}+1}=\frac{\beta_1\xi^2-1}{2\xi+1}$, where $\xi=\frac{\xi_1}{\xi_2}$. Note that in $2\frac{\xi_1}{\xi_2}+1=0$ case for that one should have $-\beta_1(\frac{\xi_1}{\xi_2})^2+1=0$ which means that 1 is root to $\beta_1-3t^2-t^3=0$- a contradiction. In this case

$$\alpha_2'=\frac{\eta_1^2\xi_2}{\Delta}\left(-\beta_1\frac{\xi_1}{\xi_2}+2\frac{\eta_2}{\eta_1}+(1+\frac{\xi_1}{\xi_2})(\frac{\eta_2}{\eta_1})^2\right)=$$

$$\frac{\eta_1^2\xi_2}{\Delta}\left(-\beta_1\xi+2\frac{\beta_1\xi^2-1}{2\xi+1}+(1+\xi)(\frac{\beta_1\xi^2-1}{2\alpha_2\xi+1})^2\right)=$$

$$\frac{\eta_1^2\xi_2}{\Delta}\frac{\beta_1^2\xi^5+\beta_1^2\xi^4-2\beta_1\xi^3-4\beta_1\xi^2-(3+\beta_1)\xi-1}{(2\xi+1)^2}=$$

$$\frac{\eta_1^2\xi_2}{\Delta}\frac{(\beta_1\xi^3-3\xi-1)(\beta_1\xi^2+\beta_1\xi+1)}{(2\xi+1)^2}.$$

$$\beta_1'=\frac{\xi_2^3}{\Delta}(\beta_1\xi^3-3\xi-1)$$

$$\alpha_4'=-\frac{\eta_1^3}{\Delta}(\beta_1-3(\frac{\eta_2}{\eta_1})^2-(\frac{\eta_2}{\eta_1})^3)=$$

$$\frac{\eta_1^3}{(2\xi+1)^3\Delta}(\beta_1^3\xi^6+6\beta_1^2\xi^5-20\beta_1\xi^3-15\beta_1\xi^2+6(1-\beta_1)\xi+2-\beta_1)$$

(in particular, this expression does not vanish),

$$\Delta=\xi_2\eta_1(\xi\frac{\beta_1\xi^2-1}{2\xi+1}-1)=\xi_2\eta_1\frac{\beta_1\xi^3-3\xi-1}{2\xi+1}$$

If $\beta_1t^2+\beta_1t+1\neq 0$ has no rational root then

$$\alpha_2'=\frac{\eta_1^2\xi_2}{\Delta}\frac{(\beta_1\xi^3-3\xi-1)(\beta_1\xi^2+\beta_1\xi+1)}{(2\xi+1)^2}=$$

$\eta_1\frac{\beta_1\xi^2+\beta_1\xi+1}{2\xi+1}$ and if $\eta_1=\frac{2\xi+1}{\beta_1\xi^2+\beta_1\xi+1}$ then $\alpha_2'=1$.

To have

$$\alpha_4'=\frac{\eta_1^3}{(2\xi+1)^3\Delta}P(\xi)=1, \text{ where}$$

$$P(t)=\beta_1^3t^6+6\beta_1^2t^5-20\beta_1t^3-15\beta_1t^2+6(1-\beta_1)t+2-\beta_1=$$
$$(\beta_1^2t^3-3t-1)(\beta_1^2t^3+6\beta_1t^2+3\beta_1t+\beta_1-2),$$

one should have $\frac{\eta_1^2\xi_2}{(2\xi+1)^2\Delta}Q(\xi)=\frac{\eta_1^3}{(2\xi+1)^3\Delta}P(\xi)$, where $Q(\xi)=(\beta_1\xi^3-3\xi-1)(\beta_1\xi^2+\beta_1\xi+1)$, that is $\xi_2=\eta_1\frac{P(\xi)}{(2\xi+1)Q(\xi)}=\frac{\beta_1^2\xi^3+6\beta_1\xi^2+3\beta_1\xi+\beta_1-2}{(\beta_1\xi^2+\beta_1\xi+1)^2}$. In this case

$$\beta_1'=\frac{\xi_2^3}{\Delta}(\beta_1\xi^3-3\xi-1)=\frac{(2\xi+1)\xi_2^2}{\eta_1}=$$

$\frac{(\beta_1^2\xi^3+6\beta_1\xi^2+3\beta_1\xi+\beta_1-2)^2}{(\beta_1\xi^2+\beta_1\xi+1)^3}$ and one comes to MSC

$$A_{10}(\mathbf{c})=\begin{pmatrix}0&1&1&1\\ \beta_1&0&0&-1\end{pmatrix}\simeq\begin{pmatrix}0&1&1&1\\ \beta_1'(a)&0&0&-1\end{pmatrix},$$

where polynomial $(\beta_1t^3-3t-1)(\beta_1t^2+\beta_1t+1)$ has no rational root, $\frac{-1}{2}\neq a\in\mathbb{F}$ and $\beta_1'(t)=\frac{(\beta_1^2t^3+6\beta_1t^2+3\beta_1t+\beta_1-2)^2}{(\beta_1t^2+\beta_1t+1)^3}$.

If $\alpha_1=\alpha_2=0$ then $\alpha_1'=\frac{1}{\Delta}\left(-\beta_1\eta_1\xi_1^2+\alpha_4\eta_2\xi_2^2\right)$,
$\alpha_2'=\frac{-1}{\Delta}\left(\beta_1\eta_1^2\xi_1-\alpha_4\eta_2^2\xi_2\right)$,
$\alpha_4'=\frac{-1}{\Delta}\left(\beta_1\eta_1^3-\alpha_4\eta_2^3\right)$, $\beta_1'=\frac{1}{\Delta}\left(\beta_1\xi_1^3-\alpha_4\xi_2^3\right)$.

In this case $\alpha_1'=\alpha_2'=0$ if and only if $\xi_1=\eta_2=0$ or $\xi_2=\eta_1=0$ and one can make $\alpha_4'=1$, $\beta_1'=\xi_1^3\beta_1^{\pm1}$ to have

$$A_{11}(p)=\begin{pmatrix}0&0&0&1\\ p&0&0&0\end{pmatrix}\simeq\begin{pmatrix}0&0&0&1\\ \overline{p}&0&0&0\end{pmatrix},$$

where $p,\overline{p}\in\mathbb{Z}$ are cube free and $p\overline{p}=q^3$ for some $q\in\mathbb{Z}$.

If polynomial $\beta_1-3\alpha_1t-3\alpha_2t^2-\alpha_4t^3$ ($\beta_1t^3-3\alpha_1t^2-3\alpha_2t-\alpha_4$) has a rational root then by making $\frac{\eta_2}{\eta_1}$ (respectively, $\frac{\eta_1}{\eta_2}$) equal to this root one can make $\alpha_4'=0$. Therefore, further it is assumed that $\alpha_4=0$.

Let us consider g with $\eta_1=0$ to have $\alpha_4'=0$. In this case $\Delta=\xi_1\eta_2$ and
$\alpha_1'=\xi_1\left(\alpha_1+2\alpha_2\frac{\xi_2}{\xi_1}\right)$,
$\alpha_2'=\alpha_2\eta_2$,
$\beta_1'=\frac{\xi_1^2}{\eta_2}\left(\beta_1-3\alpha_1\frac{\xi_2}{\xi_1}-3\alpha_2(\frac{\xi_2}{\xi_1})^2\right)$.

Case a: $\alpha_2\neq 0$. One can consider $\frac{\xi_2}{\xi_1}=\frac{-\alpha_1}{2\alpha_2}$ to get $\alpha_1'=0,\alpha_2'=1$ and $\beta_1'=\xi_1^2\frac{3\alpha_1^2+4\alpha_2\beta_1}{4}$. Therefore one can make β_1' equal to 0 or p, depending on $3\alpha_1^2+4\alpha_2\beta_1$ to have

$$A_{12}(p)=\begin{pmatrix}0&1&1&0\\ p&0&0&-1\end{pmatrix},$$

where $p=0$ or $p\in\mathbb{Z}$-square free.

Case b: $\alpha_2=0$. Then $\alpha_2'=\alpha_4'=0$ and $\alpha_1'=\xi_1\alpha_1,\beta_1'=\frac{\xi_1^2}{\eta_2}\left(\beta_1-3\alpha_1\frac{\xi_2}{\xi_1}\right)$. Therefore if $\alpha_1\neq 0$ one can make $\alpha_1'=1,\beta_1'=0$ to get $A'=\begin{pmatrix}1&0&0&0\\ 0&-1&-1&0\end{pmatrix}$, which isomorphic to $A_{12}(0)$, if $\alpha_1=0$ then $\alpha_1'=0$ and one can make $\beta_1'=1$ to come to

$$A_{13}=\begin{pmatrix}0&0&0&0\\ 1&0&0&0\end{pmatrix}.$$

Remark. *The main result of this paper was announced in [10]. After that some defects were found in A_{10} and A_{11} cases which are corrected in this paper.*

III CONCLUSION

The method used before to classify two dimensional algebras over fields, where every second and third order polynomial has a root, can be successfully adapted to classify such algebras over the field of rational numbers.

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