Acta of Turin Polytechnic University in Tashkent, 2021, 11, 27-28
Published Online March 2021 in Acta TTPU (http://www.acta.polito.uz/)

ON DYNAMICS OF SOME DIFFERENCE EQUATIONS

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Abstract—In present work we investigate the difference equations $x_{n+1} = f(x_n),\ n \geq 0$. We consider the case when the function f is power series. It is proved that under some condition to f the solution of difference equation asymptotically equivalent to $bn^{\alpha},\ \alpha > 0$, as $n \to \infty$.

Keywords— difference equation, power series, solution of difference equation, asymptotic behaviour.

I The formulation of main result and its proof

Difference equations usually describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $n+1$ st generation x_{n+1} is a function of the $n-$th generation x_n. This relation expresses itself in the **difference equation**

$$x_{n+1} = f(x_n). \tag{1}$$

We may look at this problem from another point of view. Starting from a point x_0, one may generate the sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots.$$

For convenience we adopt the notation

$$f^{(2)}(x_0) = f(f(x_0)), f^{(3)}(x_0) = f(f(f(x_0))),$$

etc. The quantity $f(x_0)$ is called the first iterate of x_0 under $f; f_2(x_0)$ is called the second iterate of x_0 under f; more generally, $f_n(x_0)$ is the $n-$th iterate of x_0 under f.

The set of all (positive) iterates $f_n(x_0) : n \geq 0$ where $f_0(x_0) = x_0$ by definition, is called the (positive) orbit of x_0 and will be denoted by $O(x_0)$. This iterative procedure is an example of a discrete dynamical system. Letting $x(n) = f_n(x_0)$, we have

$$x_{n+1} = f_{n+1}(x_0) = f(f_n(x_0)) = f(x_n),$$

and hence we recapture (1). Observe that $x(0) = f_0(x_0) = x_0$. For example, let $f(x) = x^2$ and $x_0 = 0.6$. To find the sequence of iterates $f_n(x_0)$, we key 0.6 into a calculator and then repeatedly depress the x^2 button. We obtain the numbers

$$0.6, 0.36, 0.1296, 0.01679616, \ldots.$$

A few more key strokes on the calculator will be enough to convince the reader that the iterates $f_n(0.6)$ tend to 0. After this discussion one may conclude correctly that difference equations and discrete dynamical systems represent two sides of the same coin.

In present paper we study the one-dimensional difference equation :

$$x_{n+1} = x_n + x_n(ax_n^q + a_1 x_n^{q_1} + a_1 x_n^{q_2} + \ldots),\ n \in \mathbb{N}, \tag{2}$$

where $\alpha \neq 0,\ \alpha_i \in R^1,\ i \geq 1$ and $0 < q < q_1 < q_2 < \ldots < q_n < \ldots$.

The right side of (2) is formal series. The difference equation

$$x_{n+1} = x_n - x_n^2\ \ n \in \mathbb{N}$$

which is the particular case of (2) were studied in [1-4].

We are interesting the asymptotic behaviour of the difference equation (2).

The main aim of the present paper is to prove the following result.

Theorem I.1 *The unique solution of difference equation (2)can be written in the form*

$$x_n = n^{\alpha}(b + o(1)), \tag{3}$$

where the constants $\alpha < 0,\ b \neq 0$ does not depend on n, if and only if

$$\alpha q = -1,\ \alpha = ab^q, \tag{4}$$

Proof of Theorem 1.1. Necessity. Suppose the equation (2) has the solution in the form (3) i.e.

$$x_n = n^{\alpha}(b+u_n), \tag{5}$$

where the sequence $u_n \to 0,\ n \to \infty$. Then we have

$$x_n \sim bn^{\alpha},\ n \to \infty. \tag{6}$$

Dividing both parts of equality (2) we obtain:

$$\frac{x_{n+1}}{x_n} = 1 + ax_n^q + a_1 x_n^{q_1} + a_1 x_n^{q_2} + \ldots,\ \ n \in \mathbb{N}, \tag{7}$$

Using the relation (5) we obtain, that

$$\frac{(n+1)^{\alpha}}{n^{\alpha}} = 1 + ab^q n^{q\alpha} + a_1 x_n^{q_1} + a_1 x_n^{q_2} + \ldots,\ \ n \in \mathbb{N}, \tag{8}$$

hence

$$(1+\frac{1}{n})^{\alpha} - 1 \sim ab^q n^{q\alpha},\ n \to \infty.$$

The implies the equality (3).

Sufficiency. First we consider the following equation:

$$u_{n+1} = (1-\frac{1}{n})u_n + f(n,u_n),\ n = N, N+1, \ldots.$$

The function $f(n,u)$ satisfy the following conditions:

$$|f(n,0)| \leq \frac{\tau}{n^{1+\beta}},\ \tau > 0,\ \beta > 0,$$

$$|f(n,u) - f(n,v)| \leq \frac{M(N,R)}{n}|u-v|,$$

where $n \geq N,\ |u| \leq R,\ |v| \pm R,\ M(N,R) \to 0$ as $N \to \infty,\ R \to 0$.

We need the following

Lemma I.2 *If the function $f(n,u)$ satisfy the conditions (5),(6), then any solution u_n of equation (4) with initial values from a small neighbourhood of zero tends to zero as $n \to \infty$.*

Proof of Lemma Using the method variation of constant we pass from equation (4) to following equation:

$$u_{n+1} = \frac{c}{n} + \frac{1}{n}\sum_{r=N}^{n} kf(k,u_k),$$

where C is arbitrary constant.

We show that the equation (7) can be solved by the method of successive approximations. Consider the set B_v of sequences u_n satisfying the condition

$$|u_n| \leq \frac{P_u}{n^{1-v}},$$

where $P_u,\ 0 < v < 1$. For the elements of the set B_v we define the norm by

$$\|u\| = \sup_{n \geq N} n^{1-v}|u_n|.$$

It is easy to check that the set with defined norm is Banach space. We show that the operator

$$Lu_n = \frac{c}{n} + \frac{1}{n}\sum_{r=N}^{n} kf(k,u_k)$$

translate the space B_v into B_v and it is compressive operator for v such that $0 < 1-v < \beta$ (see [2-5]).

Assume that $u_n \in B_v$. Then

$$|Lu_n| \leq \frac{|c|}{n} + \frac{1}{n}\sum_{r=N}^{n} k|f(k,u_k)| \leq$$

$$\leq \frac{|c|}{n} + \frac{1}{n}\sum_{r=N}^{n} k|f(k,0)| + \frac{1}{n}\sum_{r=N}^{n} |f(k,u_k) - f(k,0)| \leq$$

$$\leq \frac{|c|}{n} + \frac{\tau}{n}\sum_{r=N}^{n} \frac{1}{k^{\beta}} + \frac{M(N,R)}{n}\sum_{r=N}^{n} \frac{1}{k^{1-v}} \leq \frac{\widetilde{P}_u}{n^{1-v}}.$$

The last relations imply that $Lu_n \in B_v$. We have

$$\|Lu - Lv\|_v = \sup_{n \geq N} n^{1-v}\frac{1}{n}\Big|\sum_{r=N}^{n} k(f(k,u_k) - f(k,v_k)\Big| \leq$$

$$\leq \sup_{n \geq N} \frac{1}{n_v}\sum_{r=N}^{n} M(N,R)|u_k - v_k| \leq M(N,R)B\|u-v\|_v,$$

where

$$B \geq \sup_{n \geq N} \frac{1}{n_v}\sum_{r=N}^{n} \frac{1}{k^{1-v}}.$$

Hence, the operator (8) under condition $MB < 1$ is compressive and the equation (7) has a unique solution in the set B_v. Lemma I.2 is completely proved.

Assume the conditions (3) hold. Changing variables in equation (7) by (2) we get the equation of form (4) It can be easy clacked that the function $f(n,u_n)$ satisfies the conditions of lemma 1.2. The theorem 1.1 is completely proved.

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